ANALYTIC GEOMETRY
AND CALCULUS

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By

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SECOND EDITION

New York: JOHN WILEY & SONS, Inc.

London

SECOND EDITION
Tenth Printing, September, 1962

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

It is the object of this book to present analytic geometry and calculus in the form and order in which these subjects are required for courses in science and engineering. In these courses calculus is the subject that is most used. After a mere introduction of graphical representation the book therefore begins with calculus. In fact the first three chapters provide a complete elementary course in the subject, including integration as a process of summation. Since these chapters contain the analysis that is most needed, the physics courses taken simultaneously are enabled to make free use of calculus almost from the beginning. The result is a saving of time which is becoming of great importance because of the pressure to include additional subjects as well as more advanced material in the undergraduate program.

Because of the convenience of complex numbers and vectors in the treatment of alternating currents, mechanics, and the analysis of electromagnetic and fluid fields, an elementary treatment of complex numbers and vector analysis is included.

I am indebted to Professor Philip Franklin for numerous problems and to W. S. Loud for many of the answers.

H. B. Phillips

Cambridge, Mass.
February 1946

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CHAPTER I

LIMITS AND CONTINUITY

1. Real Numbers. The process of counting gives whole numbers. Division and subtraction give fractions and negative numbers. Whole numbers and fractions, whether positive or negative, are called *rational* numbers. A number, like $\sqrt{2}$, which can be expressed to any required degree of accuracy, but not exactly by a fraction, is called *irrational*. Rational and irrational numbers, whether positive or negative, are called *real*.

The *absolute value* of a real number is the number without its algebraic sign. The absolute value of x is represented by the notation $|x|$. Thus

$$|-2| = |+2| = 2.$$

2. Points on a Line. Real numbers can be represented graphically by points on a straight line. To do this choose a unit of length and take a point O on the line as origin. Upon the point O mark the number 0. On one side of O mark positive numbers, on the other negative, making the number at each point equal in absolute value to the distance from O to the point.

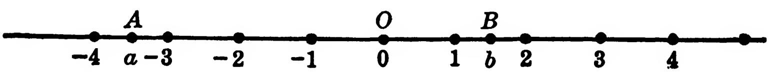

FIGURE 1.

The line with its associated numbers is called a *scale*. Proceeding along the scale in one direction (to the right in Figure 1) the numbers increase algebraically. Proceeding in the other direction the numbers decrease. The direction in which the numbers increase is called *positive*, that in which they decrease *negative*.

To each real number corresponds a point on the scale, and to two numbers a, b correspond two points A, B whose distance apart is

$$|b - a|.$$

To each relation between numbers corresponds a relation between points, and conversely. Consequently in describing relations

between numbers we often use geometrical language when the intention is obvious.

3. Function. If two variables x and y are so related that when a value is assigned to x a value of y is determined, then y is called a *function* of x.

Thus, the area of a circle is a function of its radius; for, the radius being given, the area can be calculated. The temperature at a given point is a function of the time; for, the time being given, the temperature has a definite value.

The function may not be defined for all values of the variable but only for certain values. The values which a variable can actually take constitute its *range*, and the function is said to be defined on that range. That is,

A quantity y is called a function of a quantity x on a given range, if to each value of x on that range corresponds a definite value of y.

In particular, the range consisting of all real numbers between two values a and b is called an *interval*, and a variable which takes all values in an interval or all real values is called *continuous*. The interval consisting of the numbers a, b and all real numbers between a and b is sometimes indicated by the notation (a, b).

For example, if x and y are the sides of a right triangle with hypotenuse a,

$$y = \sqrt{a^2 - x^2}.$$

Since, however, the side cannot be negative or as great as the hypotenuse, y is defined only for values of x between 0 and a. The range of the variable is the interval

$$0 < x < a,$$

and on that range y is a function of x.

More generally,

A quantity y is called a function of any number of quantities x_1, $x_2, \cdots, x_n$ on given ranges if to each set of values, $x_1, x_2, \cdots, x_n$, on those ranges corresponds a definite value of y.

Thus, the area of a triangle is a function of the lengths of its three sides. The pressure of a given mass of gas is a function of its temperature and volume.

4. Independent and Dependent Variables. In many problems there occur a number of variables connected by equations. Arbitrary values are assigned to some of these variables and the others are determined as functions. Those taking arbitrary values are

called *independent* variables. Those determined are called *dependent* variables.

Which are chosen as independent and which as dependent is often arbitrary. Certain choices may, however, be much more convenient than others. Thus if x and y are real and

$$y = x^3 + x,$$

either x or y could be taken as independent variable. If x is independent variable, values of y are, however, easy to calculate, but if y is independent the corresponding values of x are much more difficult to determine.

5. Functional Notation. A particular function of x is often represented by the notation $f(x)$, which should be read function of x, or f of x, not f times x. For example,

$$f(x) = \sqrt{x^2 + 1}$$

means that $f(x)$ is a symbol for $\sqrt{x^2 + 1}$. Similarly

$$y = f(x)$$

means that y is some definite (though perhaps unknown) function of x.

If it is necessary to consider several functions in the same discussion, they are distinguished by subscripts or accents or by the use of different letters. Thus $f_1(x)$, $f_2(x)$, $f'(x)$, $f''(x)$, $g(x)$ (read f-one of x, f-two of x, f-prime of x, f-second of x, g of x) represent (presumably) different functions of x.

Similarly, a function of two variables x and y is represented by the notation $f(x, y)$, a function of three variables x, y, z by the notation $f(x, y, z)$, etc. Thus the equations

$$u = F(x, y), \quad v = G(x, y, z)$$

express that u is a function of x and y and that v is a function of x, y, and z.

The f in the symbol for a function may often be interpreted as representing an operation performed on the variable or variables. Thus, if

$$f(x, y) = \sqrt{x^2 + y^2},$$

f represents the operation of squaring the variables, adding, and extracting the square root of the result. If x and y are replaced

by any other quantities the same operation is to be performed on those quantities. For example,

$$f(1, 2) = \sqrt{1^2 + 2^2} = \sqrt{5},$$

$$f(x, y + 1) = \sqrt{x^2 + (y + 1)^2} = \sqrt{x^2 + y^2 + 2y + 1}.$$

6. Rectangular Coördinates. Graphically a function y of a continuous variable x can be represented by a curve. One way to do this is to consider x and y as rectangular coördinates of a point in a plane.

For this purpose take two perpendicular scales $X'X$, $Y'Y$ with their zero points coincident at O (Figure 2). It is customary to draw $X'X$, called the x-axis, horizontal with positive direction to the right, and $Y'Y$, called the y-axis, vertical with positive direction upward. The point O is called the origin.

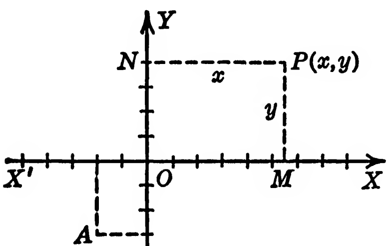

FIGURE 2.

From any point P in the plane drop perpendiculars PM, PN to the axes. Let the number at M in the scale $X'X$ be x and that at N in the scale $Y'Y$ be y. The numbers x and y are called the rectangular coördinates of the point P with respect to the axes $X'X$, $Y'Y$. The number x is called the *abscissa*, the number y the *ordinate*, of P.

The point with coördinates x, y is represented by the notation (x, y). To signify that P has the coördinates x, y the notation $P(x, y)$ is often used. For example, $(1, -2)$ is the point $x = 1$, $y = -2$. Similarly, $A(-2, -3)$ signifies that the abscissa of A is -2 and its ordinate is -3.

The process of locating a point with given coördinates is called *plotting*. To obtain the curve representing a given function

$$y = f(x)$$

we substitute values for x, calculate the corresponding values of y, and plot the resulting points (x, y). The locus of such points is the required curve. It is often called the graph of the equation $y = f(x)$.

Conversely, any locus (such as that in Figure 3) which cuts each line parallel to OY in a single point determines y as a func-

tion of x. The value of the function which corresponds to a given value x of the variable is merely the ordinate at the point of abscissa x.

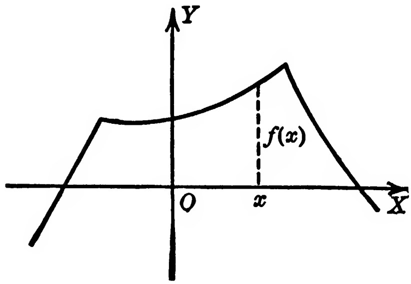

FIGURE 3.

We thus have two ways of presenting a function $f(x)$. One is to give an expression by means of which its values can be calculated. The other is to give a graph on which its values can be measured.

Example 1. Plot the curve

$$y = x^2 - x$$

for values of x between -2 and 3.

By substituting values of x and calculating the corresponding values of y we obtain the pairs of coördinates indicated in the following table:

$x =$	$-2,$	$-1,$	$0,$	$1,$	$2,$	3
$y =$	$6,$	$2,$	$0,$	$0,$	$2,$	6

We plot the points represented by these pairs of coördinates and draw a smooth curve through the resulting points. The curve is shown in Figure 4.

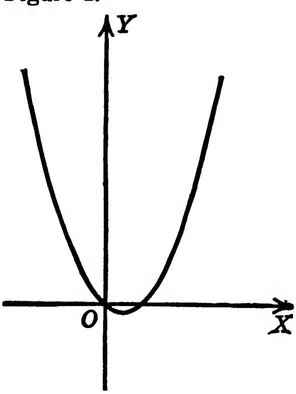

FIGURE 4.

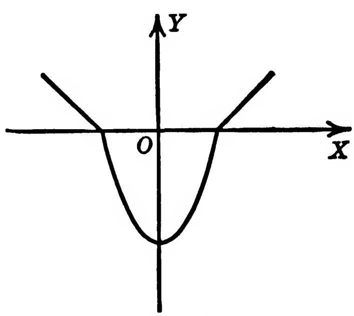

FIGURE 5.

Example 2. When x is between -2 and 2 a function has the value

$$f(x) = x^2 - 4$$

and outside that range it has the value

$$f(x) = |x| - 2.$$

Plot the graph of $y = f(x)$.

We substitute for x and calculate y, using $y = x^2 - 4$ if x is between -2 and 2, and $y = |x| - 2$ for all other values of x. Pairs of corresponding values are indicated in the following table:

$x =$	$-4,$	$-3,$	$-2,$	$-1,$	$0,$	$1,$	$2,$	$3,$	4
$y =$	$2,$	$1,$	$0,$	$-3,$	$-4,$	$-3,$	$0,$	$1,$	2

The graph is shown in Figure 5.

7. Limit of a Sequence. In many investigations direct calculation does not give the exact value of a quantity that is under consideration but does give a never-ending sequence of approximate values

$$x_1, x_2, x_3, \cdots, x_n, \cdots \tag{1}$$

which as n increases so approach a number c that the difference

$$|x_n - c|$$

ultimately becomes and remains less than any given positive number. We express this by saying that the sequence converges to c as limit, or that the variable x_n tends to the limit c as n increases.

For example, to obtain $\sqrt{2}$ we use a process which gives the value approximately as a decimal. To six places we find

$$\sqrt{2} = 1.414214.$$

If r_n is the value to n decimal places, we have an infinite sequence of approximate values

$$r_1, r_2, r_3, \cdots, r_n, \cdots$$

such that

$$|r_n - \sqrt{2}| < 10^{-n}. \tag{2}$$

As n increases, the difference $|r_n - \sqrt{2}|$ ultimately remains less than any fixed positive number. The sequence of values r_n converges to $\sqrt{2}$ as limit.

A quantity is often determined as the sum of an infinite series. For example, by division we find

$$s_n = \frac{1 - r^{n+1}}{1 - r} = 1 + r + r^2 + \cdots + r^n, \tag{3}$$

n being any positive integer. If $|r| < 1$, the number

$$\frac{r^{n+1}}{1 - r}$$

can be made as small as we please by making n sufficiently large. As n increases, s_n therefore tends to the limit

$$\frac{1}{1 - r},$$

and the limit approached by the right side of (3) is indicated by the notation

$$1 + r + r^2 + \cdots + r^n + \cdots.$$

Thus, if $|r| < 1$, we have

$$\frac{1}{1 - r} = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots. \tag{4}$$

To determine the circumference and area of a circle we inscribe a regular polygon of n sides and determine its perimeter p_n and area A_n. We thus have two sequences

$$p_3,\ p_4,\ p_5,\ \cdots,\ p_n,\ \cdots,$$

$$A_3,\ A_4,\ A_5,\ \cdots,\ A_n,\ \cdots,$$

which converge to limits p and A respectively. These limits are by definition taken as the circumference and area of the circle.

In each of these cases we have a variable (approximation to n decimal places, sum of n terms of a series, perimeter or area of a polygon of n sides) which, for the various positive integral values of n, takes an infinite sequence of values

$$x_1,\ x_2,\ \cdots,\ x_n,\ \cdots,$$

and these so cluster about a number c that, as n increases, the difference $|x_n - c|$ remains less than a given positive number ϵ from some value of n onward. If that value is $n = N$, the above definition is equivalent to the following:

An infinite sequence of numbers $x_1, x_2, x_3, \cdots, x_n, \cdots$ *is said to converge to the number* c *as limit if for each positive number* ϵ, *no matter how small, there is a positive integer* N *such that*

$$|x_n - c| < \epsilon \tag{5}$$

for all values of n *greater than* N.

This behavior of the sequence, or of the variable x_n, is indicated by the notation $x_n \to c$ or

$$\lim_{n \to \infty} x_n = c. \tag{6}$$

Example 1. Find the limit approached by the decimal

$$0.12121212\cdots$$

as the number of places is indefinitely increased.

Expressed as a series, this decimal is equivalent to

$$\frac{12}{100}\left[1 + \frac{1}{100} + \frac{1}{(100)^2} + \frac{1}{(100)^3} + \cdots\right].$$

By equation (4), its value is therefore

$$\frac{12}{100}\frac{1}{1 - \frac{1}{100}} = \frac{12}{99}.$$

Example 2. Find the value of

$$\lim_{n \to \infty} \frac{2n^2 - n + 1}{n^2 + 3}.$$

Dividing numerator and denominator by n^2, we obtain

$$\frac{2n^2 - n + 1}{n^2 + 3} = \frac{2 - \dfrac{1}{n} + \dfrac{1}{n^2}}{1 + \dfrac{3}{n^2}}.$$

As n increases, $\dfrac{1}{n}$ and $\dfrac{1}{n^2}$ tend to zero and the right side of this equation tends to the value 2. The limit is therefore 2.

Example 3. $\lim\limits_{n \to \infty} (\sqrt{n+1} - \sqrt{n})$.

Writing the given expression in the form

$$\frac{\sqrt{n+1} - \sqrt{n}}{1}$$

and rationalizing the numerator, we obtain

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

As n increases, the right side of this equation tends to zero. The limit is thus zero.

Example 4. Show that the area of a circular sector bounded by an arc s and two radii is

$$A = \tfrac{1}{2}rs. \tag{7}$$

Divide the arc into n equal parts and draw radii to the points of division. These and chords between consecutive points determine a set of n triangles having equal bases b_n and equal altitudes h_n. The sum of lengths of the chords is

$$s_n = nb_n,$$

and the sum of the areas of the triangles is

$$A_n = n(\tfrac{1}{2}h_nb_n) = \tfrac{1}{2}h_ns_n.$$

As n increases, s_n approaches the arc s as limit, h_n approaches the radius r, and A_n approaches the area of the sector. Thus

$$A = \lim_{n \to \infty} \tfrac{1}{2}h_ns_n = \tfrac{1}{2}rs,$$

which was to be proved.

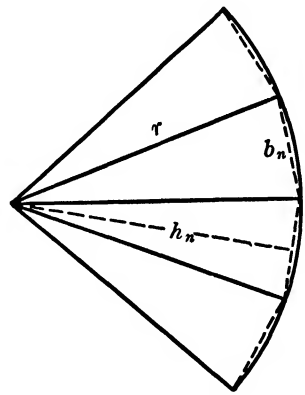

FIGURE 6.

8. Limit of a Function. Our discussion of limits has hitherto been restricted to variables that take only discrete values. Suppose now that x is a continuous variable and $f(x)$ a function defined at all points of an interval except possibly $x = c$. We are interested in values of $f(x)$ at points near $x = c$. If these differ arbitrarily little from a number l at all points sufficiently near $x = c$, we say $f(x)$ approaches, or tends to, the limit l as x tends to c. This is indicated by the notation

$$\lim_{x \to c} f(x) = l. \tag{1}$$

As an example consider the function

$$\frac{x^2 - 4}{x^2 + 4}$$

for values of x near 2. If we place $x = 2$, the function becomes zero, and, if x is near 2, the function is evidently near zero. Thus

$$\lim_{x \to 2} \frac{x^2 - 4}{x^2 + 4} = 0.$$

As a second example take

$$\frac{x^2 - 1}{x - 1}$$

for values of x near 1. If we place $x = 1$, the function assumes the form

$$\frac{0}{0},$$

which has no meaning. By division, however, we obtain

$$\frac{x^2 - 1}{x - 1} = x + 1.$$

If x is nearly equal to 1, the right side of this equation is nearly equal to 2 and the left side has the same value. Consequently

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

As a final example we take

$$\frac{\sqrt{x + 1} - 1}{x}$$

for values of x near zero. Since division by zero is meaningless, we cannot place $x = 0$. If, however, we multiply above and below by $\sqrt{x + 1} + 1$, we have

$$\begin{aligned}
\frac{\sqrt{x + 1} - 1}{x} &= \frac{(\sqrt{x + 1} - 1)(\sqrt{x + 1} + 1)}{x(\sqrt{x + 1} + 1)} \\
&= \frac{x}{x(\sqrt{x + 1} + 1)} \\
&= \frac{1}{\sqrt{x + 1} + 1}.
\end{aligned}$$

In the expression on the right we can now let x tend to zero and so obtain

$$\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} = \frac{1}{2}.$$

In each of these cases we have a function which at points near a value $x = c$ is approximately equal to a number l, the error being less than any stated amount ϵ at all points of some interval

of center $x = c$. If this interval extends a distance δ on each side of c, the above definition is equivalent to the following:

The function $f(x)$ is said to approach the limit l as x tends to c, if for each positive number ϵ there is a positive number δ such that

$$|f(x) - l| < \epsilon \tag{2}$$

at all points of the interval

$$|x - c| < \delta \tag{3}$$

except possibly $x = c$.

From this definition it is clear that if $x_1, x_2, x_3, \cdots, x_n, \cdots$ is any sequence of values different from c, but tending to c as limit,

$$\lim_{n \to \infty} f(x_n) = l,$$

for x_n will belong to the interval (3) from some value of n onward and then

$$|f(x_n) - l| < \epsilon.$$

In the above definition x has been restricted to have values different from c. The reason for this is partly that the value $x = c$ is not needed in the determination of the limit and partly that $f(c)$ may not be defined or may not have the proper value if it is defined.

9. Angle. In work that involves calculus angles are usually expressed in circular measure.

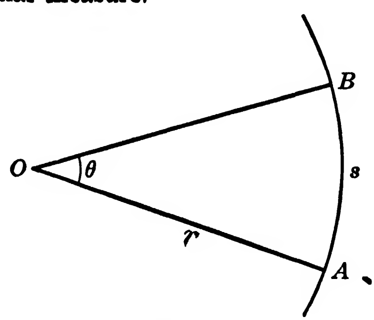

FIGURE 7.

If AOB (Figure 7) is a given angle and $AB = s$ is the length of arc it intercepts on a circle of radius r and center O, the circular measure of the angle is defined as the ratio

$$\theta = \frac{\text{arc}}{\text{rad.}} = \frac{s}{r}. \tag{1}$$

Frequently no unit is mentioned, but when it is desired to emphasize that circular measure is used we say the angle is one of θ *radians*, the radian being defined as the angle subtended by an arc equal to the radius.

When the radius r and angle θ are given, the length of the intercepted arc is determined by the equation

$$s = r\theta. \tag{2}$$

10. Limits of Trigonometric Functions. Inspection of a table of sines will show that when θ (measured in radians) is small

$$\frac{\sin\theta}{\theta}$$

is approximately equal to unity. Thus, if θ is an angle of 5°,

$$\sin\theta = .08716$$

$$\theta = .08727$$

and

$$\frac{\sin\theta}{\theta} = .9987.$$

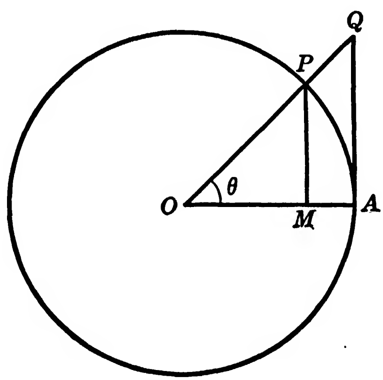

FIGURE 8.

To prove that the limit of this ratio is unity, let θ be the angle AOP at the center of a circle of unit radius (Figure 8). Since $r = 1$, equations (2), §9, and (7), §7, give

$$\text{arc } AP = \theta, \text{ sector } OAP = \tfrac{1}{2}\theta. \tag{1}$$

The inequality

$$\Delta OMP < \text{sector } OAP < \Delta OAQ \tag{2}$$

is thus equivalent to

$$\tfrac{1}{2}\sin\theta\cos\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta, \tag{3}$$

whence division by $\frac{1}{2}\sin\theta$ gives

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \tag{4}$$

and so

$$\frac{1}{\cos\theta} > \frac{\sin\theta}{\theta} > \cos\theta. \tag{5}$$

As θ tends to zero, $\cos\theta$ and $\dfrac{1}{\cos\theta}$ both tend to the limit 1, and

$$\frac{\sin\theta}{\theta}$$

is between the two. Therefore

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1. \tag{6}$$

Another limit which is frequently used is that of

$$\frac{1-\cos\theta}{\theta}$$

as θ tends to zero. By trigonometry we have

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}. \tag{7}$$

Hence

$$\frac{1-\cos\theta}{\theta} = \frac{2\sin^2\dfrac{\theta}{2}}{\theta} = \sin\frac{\theta}{2}\left(\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right). \tag{8}$$

As θ tends to zero, $\dfrac{\theta}{2}$ and $\sin\dfrac{\theta}{2}$ tend to zero and, by equation (6),

$$\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}$$

tends to the limit 1. Thus (8) gives as limit

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0. \tag{9}$$

Example 1. Find the value of

$$\lim_{x \to 0} \frac{\sin 5x}{x}.$$

By algebra we have

$$\frac{\sin 5x}{x} = 5 \frac{\sin 5x}{5x}. \tag{10}$$

As x tends to zero, $5x$ tends to zero and, by equation (6),

$$\frac{\sin 5x}{5x}$$

tends to the limit 1. From (10) we thus have

$$\lim_{x \to 0} \frac{\sin 5x}{x} = 5.$$

Example 2. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\pi - 2x}.$

Placing

$$\frac{\pi}{2} - x = y,$$

we have

$$\frac{\cos x}{\pi - 2x} = \frac{\sin y}{2y}.$$

As x tends to $\dfrac{\pi}{2}$, y tends to zero and

$$\frac{\sin y}{y}$$

to unity. Therefore

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\pi - 2x} = \lim_{y \to 0} \frac{\sin y}{2y} = \frac{1}{2}.$$

11. Properties of Limits. We often have several functions of the same variable each of which approaches a definite limit when the variable approaches a certain value. In such cases extensive use is made of certain simple properties which follow almost immediately from the definition of limit.

1. *The limit of the sum of a finite number of functions is equal to the sum of their limits.*

Suppose, for example, that X, Y, Z are three functions of the same variable x and that they approach the limits A, B, C

respectively when x approaches the limit c. The differences

$$X - A = \epsilon_1, \quad Y - B = \epsilon_2, \quad Z - C = \epsilon_3,$$

and their sum

$$(X + Y + Z) - (A + B + C) = \epsilon_1 + \epsilon_2 + \epsilon_3$$

then all tend to the limit zero as x tends to c. Thus

$$\lim (X + Y + Z) = A + B + C = \lim X + \lim Y + \lim Z,$$

which was to be proved.

2. *The limit of the product of a finite number of functions is equal to the product of their limits.*

Suppose, for example, that X, Y are two functions approaching the limits A, B respectively. Writing

$$X = A + \epsilon_1, \quad Y = B + \epsilon_2,$$

we have

$$XY - AB = A\epsilon_2 + (B + \epsilon_2)\epsilon_1.$$

Since A, B are fixed numbers and ϵ_1, ϵ_2 approach zero, the right side of this equation will ultimately become numerically less than any fixed positive number. Consequently

$$\lim XY = AB = \lim X \cdot \lim Y.$$

3. *If the limit of the denominator is not zero, the limit of the quotient of two functions is equal to the quotient of their limits.*

Using the same notation as in the preceding discussion, if B is not zero,

$$\frac{X}{Y} - \frac{A}{B} = \frac{A + \epsilon_1}{B + \epsilon_2} - \frac{A}{B} = \frac{B\epsilon_1 - A\epsilon_2}{B(B + \epsilon_2)}.$$

Since ϵ_1 and ϵ_2 approach zero and B is not zero, the limit of the right side is zero. Hence

$$\lim \frac{X}{Y} = \frac{A}{B} = \frac{\lim X}{\lim Y}.$$

If $\lim Y = 0$, the expression

$$\frac{\lim X}{\lim Y}$$

has no meaning.

12. Infinity. Sometimes a variable increases without bound. If the increase is such that, no matter how large M may be, there is a stage in the process beyond which x remains numerically larger than M, we say that x *becomes infinite*, or *tends to infinity*. If, in addition, x ultimately has a fixed algebraic sign, we write $x \to \infty$ or $x \to -\infty$ according as that sign is positive or negative.

Similarly a function $f(x)$ of a continuous variable x is said to become infinite as x tends to c if, no matter how large M may be, there is some interval of center c at all points of which (different from c) $f(x)$ is numerically greater than M. If, in addition, $f(x)$ has a fixed algebraic sign at all points of a sufficiently small interval of center c, we sometimes write

$$\lim_{x \to c} f(x) = +\infty, \quad \lim_{x \to c} f(x) = -\infty$$

according as that sign is positive or negative. These, however, are not proper limits, and theorems concerning limits may not apply to such cases.

In the definition just given it should be noted that nothing is said about the value of the function at the point $x = c$. The function could have any value at that point, but usually no value is assigned. We often say the function is infinite at the point, but this should be considered as merely a brief way of describing a certain behavior of the function in the neighborhood of the point.

Consider, for example,

$$f(x) = \frac{1}{x}.$$

When x is near zero, $f(x)$ is very large. In fact, no matter how large the positive number M may be, $f(x)$ is numerically greater than M at all points of the interval

$$-\frac{1}{M} < x < +\frac{1}{M}$$

except $x = 0$ where it is not defined. The function becomes infinite as x tends to zero.

As an example of a different behavior consider the function

$$f(x) = 2^{\frac{1}{x}}.$$

When x tends to zero through positive values this function increases indefinitely. But when x tends to zero through negative

values it tends to zero. If both positive and negative values of x are included the function neither tends to zero nor becomes infinite when $x \to 0$.

13. Continuity. If a function $f(x)$ is represented by a connected curve, its value differs arbitrarily little from $f(c)$ when x is sufficiently near c. This property is expressed by the word continuity. That is,

A function $f(x)$ is called continuous at $x = c$ if $f(c)$ has a definite value and $f(x)$ tends to $f(c)$ as limit when x tends to c.

In view of the definition of limit (§8) the condition that $f(x)$ be continuous at $x = c$ is that for each positive number ϵ there exist a positive number δ such that

$$|f(x) - f(c)| < \epsilon \tag{1}$$

at all points of the interval

$$|x - c| < \delta. \tag{2}$$

From the analogous properties of limits it follows that if $y = f(x)$ is continuous at $x = c$ any positive power of y is also continuous,

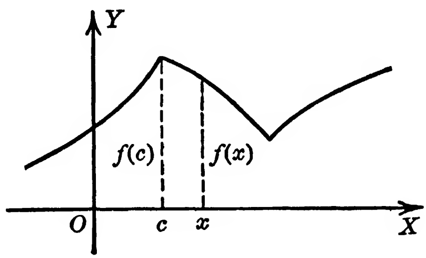

FIGURE 9.

and if two functions are continuous their sum, product, and quotient are continuous provided the denominator is not zero. By repeated application we conclude that any algebraic expression in x is continuous for values of x that do not make any denominator zero.

At a point where a function is not continuous it is called *discontinuous*. The most common discontinuities are those at which the function becomes infinite and those at which it takes a finite jump.

A simple illustration of finite jump is furnished by a *step function*. By this we mean a function which is graphically represented by a set of horizontal line segments with abrupt rise or fall from

one segment to the next, as in Figure 10. Many functions have this form because fractional values are not used. Thus the cost of first-class postage is a definite number of cents per ounce or fraction of an ounce. The cost y therefore remains constant between consecutive integral values of the weight x in ounces but changes abruptly when x passes through a whole number. Similarly, the

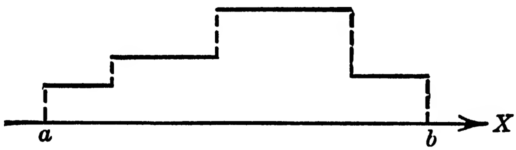

FIGURE 10.

cost of life insurance is determined by the age of the applicant at the nearest birthday. As a function of age it thus changes abruptly midway between birthdays.

At a discontinuity a function may take the upper value, the lower value, or any other. Frequently it is not defined. In the case of postage, for example, the lower value is used, and in that of life insurance no cost midway between birthdays is defined.

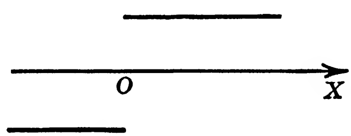

FIGURE 11.

Example 1. $f(x) = \dfrac{x}{|x|}$.

If x is positive, $|x| = x$ and $f(x) = 1$. If x is negative, $|x| = -x$ and $f(x) = -1$. The graph $y = f(x)$ is shown in Figure 11. As x passes through zero from negative to positive values the function jumps from -1 to $+1$. No value of the function is defined at $x = 0$.

Example 2. $f(x) = \dfrac{1}{x - 1}$.

As x tends to 1, $f(x)$ increases without bound, being positive if $x > 1$ and negative if $x < 1$. The function has an infinite discontinuity at $x = 1$ (Figure 12).

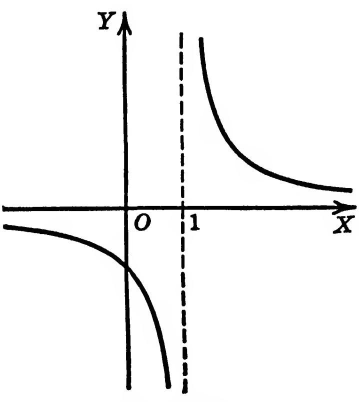

FIGURE 12.

Example 3. $f(x) = 2^{\frac{1}{x}}$.

As x tends to zero through positive values, $f(x)$ increases without bound. As x tends to zero through negative values, $f(x)$ tends to zero. There is an infinite discontinuity at $x = 0$ (Figure 13).

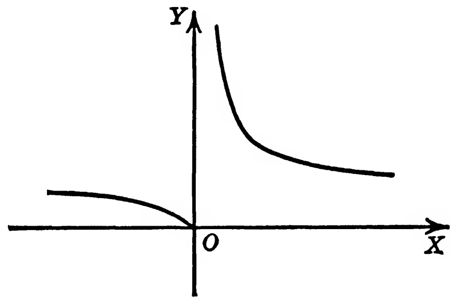

FIGURE 13.

14. Some Fundamental Theorems. In the development of calculus certain nearly obvious results are needed exact proof of which requires a more careful analysis of the nature of number than is appropriate in a textbook of this kind. For convenience of reference we state these results here, referring those who wish proofs to books on advanced calculus or function theory.

(1) **General Convergence Condition.** *A necessary and sufficient condition that the sequence*

$$x_1, x_2, x_3, \cdots, x_n, \cdots$$

converge is that for each positive number ϵ *there exist a positive integer* N *such that*

$$|x_m - x_n| < \epsilon \tag{1}$$

for all values of m *and* n *greater than* N.

This amounts to saying that the condition for convergence is that there be a place in the sequence beyond which the difference of any two terms is in numerical value less than any preassigned positive number ϵ. That a convergent sequence has this property is evident since there is a place in such a sequence beyond which all terms differ from the limit by less than $\frac{1}{2}\epsilon$ and so differ from each other by less than ϵ. What is important is that the converse is also true, namely, that, if for each positive ϵ there is a place in

the sequence beyond which the terms all differ from each other by less than ϵ, there is a limit to which the sequence converges.

When we do not have an explicit expression for the limit this theorem is the basis for inferring its existence. In the case, for example, of regular polygons inscribed in a circle we know that the areas A_n of these polygons differ arbitrarily little when n is sufficiently large, and from this we infer that A_n tends to a limit as n increases.

(2) **Continual Increase or Decrease.** *A variable approaches a limit if it always increases but never becomes greater than some fixed number or if it always decreases but never becomes less than some fixed number.*

A variable that always increases or always decreases thus either approaches a limit or becomes infinite. An illustration is the decimal representation of an irrational number. If r_n is the value to n places, this increases as n increases but never exceeds the next larger integer. It thus approaches a limit.

(3) **Extreme Value Theorem.** *If a function $f(x)$ is continuous in the interval $a \leqq x \leqq b$, it has a greatest and a least value in that interval.*

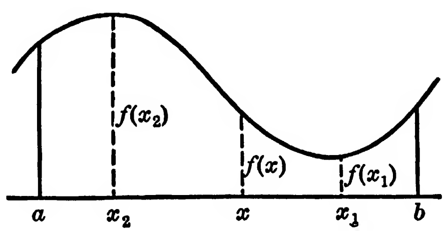

FIGURE 14.

By this statement we mean that there are numbers x_1, x_2 within the interval or at its ends such that

$$f(x_1) \leqq f(x) \leqq f(x_2) \tag{2}$$

for all values of x in the interval. Graphically this means that a finite portion of a continuous curve has a highest and a lowest point.

(4) **Intermediate Value Theorem.** *If $f(x)$ is continuous in the interval $a \leqq x \leqq b$, and N is any number between $f(a)$ and $f(b)$, there is some value c between a and b such that $f(c) = N$.*

This amounts to saying that between two given values a continuous function takes all intermediate values.

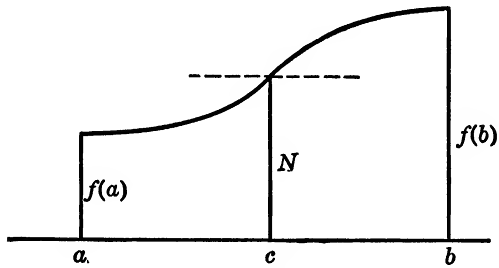

FIGURE 15.

(5) **Uniform Continuity.** *If $f(x)$ is continuous in the interval $a \leqq x \leqq b$, then for each positive number ϵ there exists a positive number δ such that*

$$|f(x_2) - f(x_1)| < \epsilon \tag{3}$$

for every pair of numbers x_1, x_2 which belong to the interval and satisfy the condition

$$|x_2 - x_1| < \delta. \tag{4}$$

Graphically this theorem states that a function continuous at all points of an interval

$$a \leqq x \leqq b$$

can be approximated by a step function in such a way that the error is not greater than ϵ at any point of the interval. To construct such a step function it is sufficient to divide the interval into parts each of length less than δ and in each part replace $f(x)$ by the value it has at any selected point of that part.

PROBLEMS

1. Show that $|xy| = |x| \cdot |y|$.

2. If x and y are both positive or both negative and $|x| \geqq |y|$, show that

$$|x + y| = |x| + |y|,$$

$$|x - y| = |x| - |y|.$$

3. If x and y have different algebraic signs, show that

$$|x + y| < |x| + |y|,$$

$$|x - y| > |x| - |y|.$$

4. Express the area A and circumference C of a circle as functions of the radius r. By eliminating r express A as a function of C.

5. Express the volume of a sphere as a function of its area of surface.

6. From the definition of function show that a constant is a function of any variable.

7. If z is a function of y and y is a function of x, show that z is a function of x.

8. Determine the interval formed by values of x that make $y = 4 - x^2$ positive.

9. Determine the range of the variable x if $|x - 1| < 3$.

10. If x and y are real variables and

$$y = \frac{x^2}{1 + x^2},$$

express x as a function of y, and determine the range of values y for which it is defined.

11. If x and y are real numbers and $y^2 - 2y + x = 0$, express y as a function of x, and determine the range of values of x for which it is defined.

12. If $f(x) = x^2 - x$, find $f(0), f(1), f(2), f(-2)$.

13. If $f(y) = 2^y$, find $f(0), f(1), f(2), f(-2)$.

14. If

$$f(x, y) = \frac{x - y}{x + y},$$

find $f(1, 0) + f(0, 1) + f(1, 1)$.

15. If $f(x) = 2 - x$, find $f(x) + 2$ and $f(x + 2)$.

16. If

$$f(x) = \frac{x - 2}{x + 1},$$

find

$$f\left(\frac{1}{x}\right) \quad \text{and} \quad \frac{1}{f(x)}.$$

17. If $f(x) = x + 1$, find $f(x^2)$ and $[f(x)]^2$.

18. If $f(x) = x + \frac{1}{x}$, show that

$$f\left(\frac{1}{x}\right) = f(x).$$

19. If

$$f(x) = \frac{ax + b}{x - a}$$

show that $f(f(x)) = x$.

20. If $f(x) = \log x$, show that $f(xy) = f(x) + f(y)$.

21. If $f(x) = a^x$, show that $f(x + y) = f(x)\,f(y)$.

22. Plot the graph of $y = x - 2$ for values of x between -1 and 2.

23. Plot the graph of $y = x^2 + x + 1$ for values of x between -2 and 1.

24. Plot the graph of $y = x^3 - x$ for values of x on the range $-2 \leqq x \leqq 2$.

25. Plot the graph of $y = x + |x|$ for values of x on the range $-1 \leqq x \leqq 1$.

26. When x is between -1 and $+1$ a function has the value $f(x) = 1 - |x|$, and outside that range it has the value $f(x) = x^2 - 1$. Plot the graph of $y = f(x)$.

27. When x is between 0 and 2 a function has the value $f(x) = x^3$ and outside that range it has the value $f(x) = 4x$. Plot the graph of $y = f(x)$.

Show that the following sequences converge, and determine their limits:

28. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \dfrac{n}{n+1}, \cdots.$

29. $1, -\frac{1}{2}, \frac{1}{3}, \cdots, \dfrac{(-1)^{n+1}}{n}, \cdots.$

30. $1, 1, 1, \cdots, 1, \cdots.$

31. $\frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, \cdots, (\frac{1}{2})^n, \cdots.$

Find the values approached by the following decimals as the number of places is indefinitely increased:

32. $1.11111\cdots.$ **33.** $3.131313\cdots.$ **34.** $2.010101\cdots.$

Determine the following limits, n being a positive integer:

35. $\lim\limits_{n \to \infty} \dfrac{n-1}{n+2}.$

36. $\lim\limits_{n \to \infty} \dfrac{n+1}{n^2+3}.$

37. $\lim\limits_{n \to \infty} \dfrac{1-n^2}{1+n^2}.$

38. $\lim\limits_{n \to \infty} \dfrac{1+n+n^3}{2n^3}.$

39. $\lim\limits_{n \to \infty} 2^{\frac{1}{n}}.$

40. $\lim\limits_{n \to \infty} [\log(n+1) - \log n].$

41. $\lim\limits_{n \to \infty} [\sqrt{2n+1} - \sqrt{2n-1}\,].$

42. $\lim\limits_{n \to \infty} [\sqrt{(n+1)^2+1} - \sqrt{n^2+1}\,].$

43. If $|x| < 1$, find the limit approached by the product

$$f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\cdots$$

as the number of factors is indefinitely increased. To do this form the product

$$(1-x)f(x) = (1-x)(1+x)(1+x^2)(1+x^4)\cdots,$$

combine the first two factors on the right, then multiply by the third factor, etc., and finally solve for $f(x)$.

44. By factoring each term and then combining consecutive terms determine the value of the infinite product

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)\cdots.$$

45. Let A_n be the area and p the perimeter of a regular polygon of n sides. If n increases and p remains constant, find

$$\lim_{n \to \infty} A_n.$$

46. The line AB of length l is divided into n equal parts and on these parts equilateral triangles are constructed as in Figure 16. If S_n is the sum of the perimeters and A_n the sum of the areas of these triangles, find

$$\lim_{n \to \infty} S_n, \quad \lim_{n \to \infty} A_n.$$

FIGURE 16.

47. In Figure 17 the triangles ABC, CDE, etc., are all equilateral, $\angle AOP = \theta$, and $AO = a$. Find the total length of the polygonal line $ABCDE\cdots$ extending from A to O.

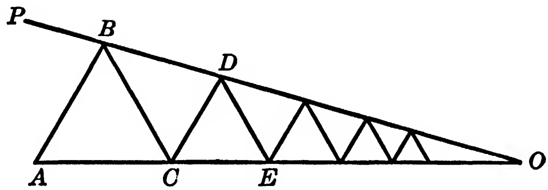

FIGURE 17.

48. Parts are cut from a line of length l in the following way. First take $\frac{1}{4}$ of the line, then $\frac{1}{4}$ of what is left, then $\frac{1}{4}$ of what is left, etc. What is the sum of all these parts?

49. A set of n circular cylinders with equal altitudes is inscribed in a cone as shown in Figure 18. When the number of cylinders increases, their alti-

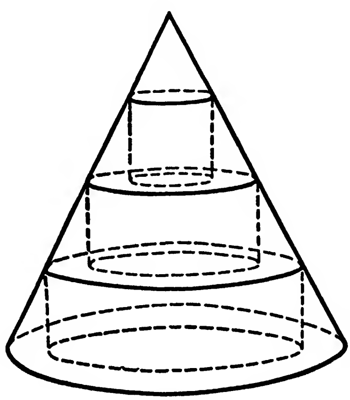

FIGURE 18.

tudes tending to zero, does the sum of their volume approach that of the cone as limit? Does the sum of the lateral areas of the cylinders approach that of the cone as limit?

50. A cone with vertex at the center of a sphere of radius r intercepts an area S on the surface of the sphere. By considering S as limit of a sum of polygons and the volume V within the cone and sphere as limit of the sum of pyramids formed by joining these polygons to the center, show that $V = \frac{1}{3}rS$.

51. By expressing each term as a power of 2 show that the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}},\ \cdots$$

converges, and find its limit.

52. Assume that a ball dropped upon the floor bounces to two-thirds the height from which it is dropped. If dropped from a height h find the total distance it moves in coming to rest.

53. If the time required to fall or rebound x ft. is $\frac{1}{4}\sqrt{x}$ sec. and the ball in the preceding problem is dropped from a height of 16 ft., find the time required to come to rest.

Determine whether the following functions tend to definite limits, and, if they do, find the values:

54. $\displaystyle\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$.

55. $\displaystyle\lim_{x \to 3} \frac{x^3 + 1}{x + 1}$.

56. $\displaystyle\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$.

57. $\displaystyle\lim_{x \to -1} \frac{x^4 - 2x - 3}{x + 1}$.

58. $\displaystyle\lim_{x \to \infty} \frac{3x + 4}{2x - 3}$.

59. $\displaystyle\lim_{x \to \infty} \frac{x^{\frac{1}{2}} - 1}{x + 2}$.

60. $\displaystyle\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$.

61. $\displaystyle\lim_{x \to \infty} \frac{x^{\frac{1}{3}} + 3}{x^{\frac{1}{3}} + 2x^{\frac{1}{6}}}$.

62. $\displaystyle\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x}$.

63. $\displaystyle\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x - 3}$.

64. $\displaystyle\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$.

65. $\displaystyle\lim_{x \to 0} 2^x$.

66. $\displaystyle\lim_{x \to 0} 2^{-\frac{1}{x^2}}$.

67. $\displaystyle\lim_{x \to 1} |x - 1|$.

68. $\displaystyle\lim_{x \to 1} \frac{|x - 1|}{x - 1}$.

69. $\displaystyle\lim_{x \to 0} \frac{\sin 3x}{x}$.

70. $\displaystyle\lim_{x \to 0} \frac{\tan x}{x}$.

71. $\displaystyle\lim_{\theta \to 0} (\theta \cos \theta)$.

72. $\displaystyle\lim_{\theta \to 0} (\theta \csc \theta)$.

73. $\displaystyle\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2}$.

74. $\displaystyle\lim_{\theta \to 0} \frac{\sin 2\theta}{\sin \theta}$.

75. $\displaystyle\lim_{\theta \to 0} \frac{\sin 3\theta}{\sin 4\theta}$.

76. $\displaystyle\lim_{x \to 0} \frac{x - \sin x}{x}$.

77. $\displaystyle\lim_{x \to \pi} \frac{\sin x}{\pi - x}$.

78. $\lim\limits_{x \to \infty} \sin x.$

79. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{\sin^2 x}.$

80. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{2x - \pi}.$

81. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1 - \sin x}{\cos x}.$

82. $\lim\limits_{x \to 0} (\csc x - \cot x).$

83. $\lim\limits_{\theta \to \frac{\pi}{4}} \dfrac{\sin \theta - \cos \theta}{\theta - \dfrac{\pi}{4}}.$

84. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin x - \cos x}{\tan x}.$

85. $\lim\limits_{\theta \to \frac{\pi}{4}} \dfrac{\sec^2 \theta - 2}{\tan \theta - 1}.$

86. By writing

$$\frac{\sin x}{\sqrt{x}} = \sqrt{x}\,\frac{\sin x}{x},$$

find

$$\lim_{x \to 0} \frac{\sin x}{\sqrt{x}}.$$

87. Considering that the logarithm of a large number is approximately equal to the number of its digits, determine

$$\lim_{x \to \infty} \frac{\log x}{x}.$$

88. Considering that the logarithm of a small number is approximately equal to the number of zeros at the beginning of its decimal expression, find

$$\lim_{x \to 0} x \log x.$$

89. By taking logarithms of both sides and using the result obtained in Problem 87, find the limit approached by $y = x^{\frac{1}{x}}$ as x increases without limit.

90. By taking logarithms of both sides and using Problem 88, find the limit approached by $y = x^x$ when x tends to zero.

Determine the values of x, if any, for which the following functions are discontinuous:

91. $f(x) = \dfrac{1}{x(x + 1)}.$

92. $f(x) = \dfrac{x^2 + 1}{x^3 + 1}.$

93. $f(x) = 2^{-\frac{1}{x}}.$

94. $f(x) = 2^{-\frac{1}{x^2}}.$

95. $f(x) = \sec x.$

96. $f(x) = \tan 2x.$

97. $f(x) = \dfrac{x - |x|}{x + |x|}.$

CHAPTER II

DERIVATIVE AND DIFFERENTIAL

15. Speed. Consider a particle moving in a straight or a curved path. If we take the distance it moves during any interval of time and divide by the time, the resulting ratio

$$\frac{\text{distance moved}}{\text{time}}$$

is called its *average speed* during that interval.

Thus, if an automobile travels 40 miles in 2 hours its average speed during that period is

$$\frac{40}{2} = 20 \text{ miles per hour.}$$

Let the time during which the motion is observed be made shorter and shorter. As the interval tends to zero the average speed during that interval usually tends to a limit. The speed *at a particular instant* is defined as the limit approached by the average speed when the interval of time measured from that instant tends to zero. That is,

$$\text{instantaneous speed} = \lim_{\text{time} \to 0} \left(\frac{\text{distance moved}}{\text{time}} \right).$$

Thus, if s_1 is the distance along the path from a fixed to a moving point at time t_1, and s_2 is the distance at a later time t_2, then $s_2 - s_1$ is the distance it travels during the interval $t_2 - t_1$ and

$$\frac{s_2 - s_1}{t_2 - t_1}$$

is its average speed during that interval. The speed at time t_1 is

$$v_1 = \lim_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1}. \tag{1}$$

For example, if a body starts from rest and moves under the action of gravity, the distance it falls in time t is

$$s = \tfrac{1}{2}gt^2 \tag{2}$$

where g, the acceleration of gravity, has approximately the value

$$g = 32.2 \text{ ft./sec.}^2$$

At time t_1 the distance fallen is

$$s_1 = \tfrac{1}{2}gt_1^2$$

and at time t_2 it is

$$s_2 = \tfrac{1}{2}gt_2^2.$$

During the interval from t_1 to t_2 the average speed is

$$\frac{s_2 - s_1}{t_2 - t_1} = \frac{\frac{1}{2}g(t_2^2 - t_1^2)}{t_2 - t_1} = \tfrac{1}{2}g(t_2 + t_1).$$

The speed at time t_1 is therefore

$$v_1 = \lim_{t_2 \to t_1} \frac{s_2 - s_1}{t_2 - t_1} = gt_1. \tag{3}$$

Since t_1 may have any value we can replace it by t, thus obtaining

$$v = gt$$

as the speed at time t.

16. Slope. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be two points on a line and ϕ the angle from the positive end of the x-axis to the line. In moving from P_1 to P_2 the changes in x and y are

$$P_1R = x_2 - x_1, \quad RP_2 = y_2 - y_1. \tag{1}$$

The ratio of these changes

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{RP_2}{P_1R} = \tan\phi \tag{2}$$

is called the *slope* of the line.

Since it is equal to $\tan\phi$ the slope is evidently independent of the positions of P_1 and P_2 on the line. If the coördinate axes have the usual positions (x-axis horizontal with positive direction to the right, y-axis vertical with positive direction upward) the slope

of a line measures its steepness and is positive (Figure 19) or negative (Figure 20) according as the right end of the line extends upward or downward.

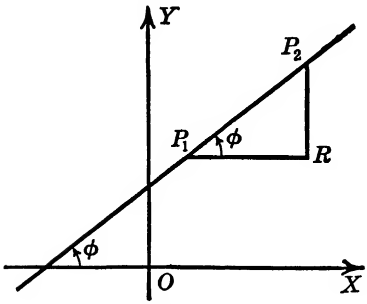

FIGURE 19.

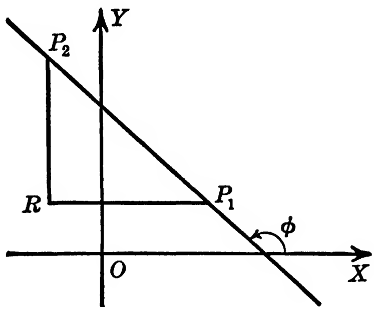

FIGURE 20.

If $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are two points on a curve (Figure 21)

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{RP_2}{P_1R}$$

is the slope of the chord P_1P_2. Keeping P_1 fixed, let P_2 move along the curve toward P_1. If the slope of the chord tends to a limit, that limit

$$m_1 = \lim_{x_2 \to x_1} \frac{y_2 - y_1}{x_2 - x_1} \tag{3}$$

is called the *slope of the curve* at P_1.

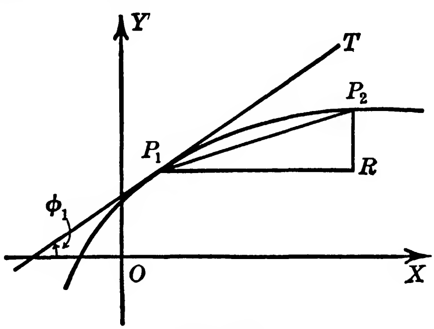

FIGURE 21.

The line P_1T with slope

$$\tan \phi_1 = m_1 \tag{4}$$

is called the *tangent to the curve* at P_1. The angle ϕ_1 is measured

from the positive end of the x-axis to the tangent, being considered positive when drawn in the direction that carries OX into OY by rotation through 90°.

Example. Find the slope of the curve

$$y = 4x - x^2$$

at the point $A(1, 3)$.

Let $P(x, y)$ be a variable point on the curve. The slope of the secant AP is

$$\frac{y - 3}{x - 1}.$$

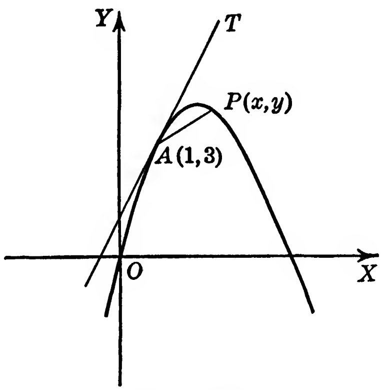

FIGURE 22.

Replacing y by its value $4x - x^2$ from the equation of the curve, this can be written

$$\frac{y - 3}{x - 1} = \frac{4x - x^2 - 3}{x - 1} = 3 - x.$$

When P moves along the curve toward A, x approaches 1 and the right side of this equation approaches 2 as limit. The slope of the curve at $A(1, 3)$ is therefore 2. The line AT with this slope is tangent to the curve at A.

17. Derivative. To obtain the speed of a moving particle (§15) we determine its distance s along the path from a fixed point as a function of the time, divide the change in s by the corresponding change in t, and take the limit as these changes tend to zero. To find the slope of a curve (§16) we express the ordinate y as a function of the abscissa x, divide the change in y by the corresponding change in x, and take the limit.

When a variable changes value, the algebraic increase (new value minus old) is called its *increment* and is represented by the symbol Δ written before the variable.

Thus if x changes from 2 to 4 its increment is

$$\Delta x = 4 - 2 = 2.$$

When x changes from 2 to -1

$$\Delta x = -1 - 2 = -3.$$

The increment is positive when there is an algebraic increase, negative when there is a decrease.

Let

$$y = f(x). \tag{1}$$

When x changes, y will also change, the new values being $x + \Delta x$, and $y + \Delta y$. These new values satisfy the equation

$$y + \Delta y = f(x + \Delta x). \tag{2}$$

By subtracting the preceding equation we obtain

$$\Delta y = f(x + \Delta x) - f(x). \tag{3}$$

The increment of y is thus a function of x and Δx.

If the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4}$$

tends to a limit when Δx tends to zero, that limit is called the *derivative of y with respect to x*. It is a function of x often represented by the notation $f'(x)$. Thus

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{5}$$

In determining this limit x is held constant and Δx varies. If for a particular value of x the limit exists, $f(x)$ is said to have a derivative, or to be *differentiable*, at that value of x.

Our definitions of speed and slope are now equivalent to the following:

If $s = f(t)$ is the distance along the path from a fixed point to a moving particle at time t, then $f'(t)$ is the speed of the particle at time t; if $y = f(x)$ is the equation of a curve, $f'(x)$ is the slope of the curve at the point of abscissa x.

To obtain the derivative of a given function $f(x)$ it is merely necessary to perform the operations indicated in equation (5).

Example 1. $f(x) = x^2$.
Replacing x by $x + \Delta x$ we obtain

$$f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\,\Delta x + (\Delta x)^2,$$

whence by subtraction

$$f(x + \Delta x) - f(x) = 2x\,\Delta x + (\Delta x)^2$$

and so

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x.$$

When Δx approaches zero this gives as limit

$$f'(x) = 2x.$$

Example 2. $f(x) = \dfrac{1}{x}\cdot$
The increment of this function is

$$f(x + \Delta x) - f(x) = \frac{1}{x + \Delta x} - \frac{1}{x} = -\frac{\Delta x}{x(x + \Delta x)}$$

whence

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = -\frac{1}{x(x + \Delta x)}\cdot$$

As Δx tends to zero this tends to the limit

$$f'(x) = -\frac{1}{x^2}\cdot$$

Example 3. Find the slope of the curve

$$y = \sqrt{x}$$

at the point (x, y).
Writing $f(x) = \sqrt{x}$, we have

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}\cdot$$

If x is not zero this gives

$$f'(x) = \frac{1}{2\sqrt{x}}$$

as the slope at the point (x, y).
Example 4. The distance traversed by a moving point at time t is

$$s = 4t^2 + 6t.$$

Find its speed.

Writing

$$f(t) = 4t^2 + 6t,$$

we have

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = 8t + 4\,\Delta t + 6.$$

When Δt tends to zero this tends to the limit

$$f'(t) = 8t + 6$$

which is therefore the speed at time t.

18. Differential. It is customary to express the derivative of a function as the ratio of two quantities called differentials. To define these we must distinguish independent and dependent variables (§4).

Let x be the independent variable and $y = f(x)$ be a function with derivative $f'(x)$. The differentials of x and y are then defined by the equations

$$dx = \Delta x, \tag{1}$$

$$dy = f'(x)\,\Delta x. \tag{2}$$

That is,

The differential of the independent variable is equal to its increment, and the differential of a function of the independent variable is equal to the product of its derivative and the increment of the independent variable.

From equations (1) and (2) we have

$$dy = f'(x)\,dx. \tag{3}$$

This fundamental equation has been obtained on the assumption that x is the independent variable. It is also valid, however, when x is not the independent variable.

To show this let t be the independent variable and

$$y = f(x), \quad x = \phi(t).$$

Since x is a function of t, $f(x)$ is also a function of t, say

$$y = F(t).$$

When values are assigned to t and Δt, values of x, y, Δx, and Δy are determined. For the values considered we assume that $f(x)$, $\phi(t)$, and $F(t)$ all have derivatives. When Δt tends to zero the equation

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta t}$$

then gives as limit

$$F'(t) = f'(x)\,\phi'(t),$$

whence

$$F'(t)\,\Delta t = f'(x)\,\phi'(t)\,\Delta t.$$

Since t is the independent variable this is equivalent to

$$dy = f'(x)\,dx,$$

which was to be proved.

If dx is not zero we can divide by dx and so obtain

$$\frac{dy}{dx} = f'(x).$$

We thus conclude that

If dx is not zero the derivative of y with respect to x is equal to the quotient dy divided by dx.

Because of the convenience of this notation we shall nearly always express derivatives in this quotient form.

Suppose, for example,

$$x = t^2, \quad y = t^2 + t.$$

Eliminating t we get

$$y = x + \sqrt{x},$$

from which we could find

$$\frac{dy}{dx} = 1 + \frac{1}{2\sqrt{x}}.$$

Instead of doing this we can, however, first determine

$$dx = 2t\,dt, \quad dy = (2t + 1)\,dt$$

and then by division obtain

$$\frac{dy}{dx} = \frac{2t+1}{2t} = 1 + \frac{1}{2t} = 1 + \frac{1}{2\sqrt{x}}.$$

19. Significance of Differential. If

$$s = f(t)$$

is the distance a body moves in time t, its differential

$$ds = f'(t)\,dt$$

is the distance it would move in the time dt if during that interval it continued to move with the speed

$$v = f'(t)$$

it has at time t. In general the speed does not remain constant, and so the distance Δs *it does move* is different from ds. But if the interval is short the speed is nearly constant and Δs is nearly equal to ds.

Similarly, if

$$y = f(x)$$

is any function,

$$dy = f'(x)\,dx$$

is the amount y would change while x is changing from x to $x + dx$ if during that interval the ratio of the changes of the two variables remained constantly equal to its limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x).$$

In general this ratio is not constant, and so Δy (the actual change in y) is different from dy. But when the changes are small this ratio is nearly constant and Δy is nearly equal to dy.

Graphically, $y = f(x)$ is the equation of a curve with slope

$$f'(x) = \tan \phi$$

at the point $P(x, y)$. In passing from $P(x, y)$ to a second point $Q(x + \Delta x, y + \Delta y)$ on the curve (Figure 23) the increments of x and y are

$$\Delta x = PR, \quad \Delta y = RQ.$$

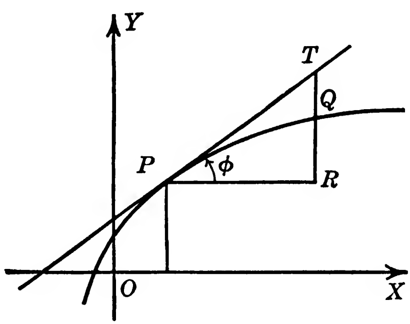

FIGURE 23.

If x is taken as independent variable,

$$dx = \Delta x = PR,$$

$$dy = f'(x)\,\Delta x = (\tan \phi)\,\Delta x = RT.$$

Thus Δx and Δy are the increments of x and y when we move from $P(x, y)$ to a second point $Q(x + \Delta x, y + \Delta y)$ on the curve, whereas dx and dy are the increments when we move from $P(x, y)$ to a point $T(x + dx, y + dy)$ on the tangent through P. The equation

$$\frac{dy}{dx} = f'(x)$$

merely expresses the obvious fact that the ratio of the sides of the right triangle PRT is equal to the tangent of the base angle.

20. Approximate Value of the Increment. We have just seen that when the values are small the increment and differential of a function are approximately equal. To investigate the nature of this approximation let x be the independent variable and $y = f(x)$ a function with differential

$$dy = f'(x)\,\Delta x. \tag{1}$$

Define a number ϵ by the equation

$$\epsilon = \frac{\Delta y}{\Delta x} - f'(x). \tag{2}$$

Then

$$\Delta y = f'(x)\,\Delta x + \epsilon\,\Delta x = dy + \epsilon\,\Delta x. \tag{3}$$

Since

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

equation (2) shows that when Δx tends to zero ϵ tends to zero, and so

$$\Delta y - dy = \epsilon\,\Delta x, \tag{4}$$

ultimately becomes as small a fraction of Δx as you please.

A variable (such as Δx) which is approaching zero as limit is called an *infinitesimal*. Such variables, though ultimately as small as you please, may have widely different relative sizes. If α and β are infinitesimals and

$$\lim_{\alpha \to 0} \frac{\beta}{\alpha} = 0,$$

β is said to be of *higher order* than α. Both infinitesimals tend to zero, but the one of higher order ultimately becomes indefinitely smaller.

Equation (4) is thus equivalent to the following:

At a value x where $y = f(x)$ has a finite derivative, Δy differs from dy by an infinitesimal of higher order than Δx.

To illustrate this numerically consider the function

$$y = f(x) = x^2.$$

The derivative is

$$f'(x) = 2x.$$

When x changes from 2 to 2.1 the differential of y is

$$dy = 2x\,\Delta x = 2(2)(0.1) = 0.40$$

and the increment is

$$\Delta y = (2.1)^2 - 2^2 = 0.41.$$

The difference is thus

$$\Delta y - dy = 0.01 = \tfrac{1}{10}\,\Delta x.$$

Similarly, when x changes from 2 to 2.01 the differential and increment have the values

$$dy = 2x\,\Delta x = 2(2)(0.01) = 0.04,$$

$$\Delta y = (2.01)^2 - 2^2 = 0.0401.$$

In this case the difference is

$$\Delta y - dy = 0.0001 = \tfrac{1}{100}\,\Delta x.$$

As Δx decreases, $\Delta y - dy$ becomes a smaller and smaller, in fact an infinitesimal, fraction of Δx.

The differential is usually simpler than the increment. When it gives a sufficiently accurate approximation it is therefore frequently used in place of the increment.

21. Differentiation of Algebraic Functions. The process of determining derivatives or differentials is called *differentiation*. We have hitherto obtained these by direct application of the limit process. Instead of this direct method differentiation is usually performed by means of formulas derived by that method.

The following formulas are sufficient for the differentiation of functional combinations formed by the operations of ordinary

algebra. In these, c, n are constants and u, v differentiable functions of a single independent variable.

I. $dc = 0.$

II. $d(u + v) = du + dv.$

III. $d(cu) = c\,du.$

IV. $d(uv) = u\,dv + v\,du.$

V. $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}.$

VI. $d(u^n) = nu^{n-1}\,du.$

These formulas give the differentials of the various functions. If the derivative with respect to a variable x is wanted it is merely necessary to divide both sides by dx. Thus from IV we get

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Formula II states that the differential of a sum can be obtained by differentiating its separate terms and adding the results. An operation that has this property is called *distributive*. This is a most important property for it permits us to break a complicated expression into simpler parts and operate separately on those parts.

Formula III states that differentiation and multiplication by a constant are *commutative* operations. By this we mean that the result is independent of the order in which the operations are performed.

To prove any one of these formulas it is convenient to represent the expression to be differentiated by a single letter. To determine the derivative with respect to an independent variable x we calculate the increment of the expression, divide by Δx, and take the limit. To reduce the result to differential form we then multiply by dx.

The only one of these proofs involving features different from those already encountered in problems is the last. To prove this let

$$y = u^n. \tag{1}$$

We must consider three cases according as n is a positive whole number, a positive fraction, or a negative rational number.

First suppose that n is a positive whole number. Let x be the independent variable. When x changes to $x + \Delta x$ the functions

u, y change to $u + \Delta u$, $y + \Delta y$, these latter values satisfying the equation

$$y + \Delta y = (u + \Delta u)^n.$$

Expanding by the binomial theorem, this becomes

$$y + \Delta y = u^n + nu^{n-1}\,\Delta u + \frac{n(n-1)}{2!}\,u^{n-2}\,(\Delta u)^2 + \cdots.$$

Since $y = u^n$,

$$\Delta y = nu^{n-1}\,\Delta u + \frac{n(n-1)}{2!}\,u^{n-2}\,(\Delta u)^2 + \cdots,$$

whence

$$\frac{\Delta y}{\Delta x} = \left[nu^{n-1} + \frac{n(n-1)}{2!}\,u^{n-2}\,\Delta u + \cdots\right]\frac{\Delta u}{\Delta x}.$$

For the values under consideration u is assumed to have a derivative with respect to x. Consequently Δu tends to zero when Δx tends to zero. Since each term in the bracket except the first contains Δu as factor, the above equation gives as limit

$$\frac{dy}{dx} = nu^{n-1}\,\frac{du}{dx}.$$

Multiplying by dx we thus have

$$dy = nu^{n-1}\,du \tag{2}$$

which was to be proved.

Next suppose that n is a positive fraction $\frac{p}{q}$ and

$$y = u^n = u^{\frac{p}{q}}. \tag{3}$$

Then

$$y^q = u^p. \tag{4}$$

Since p and q are positive integers we can apply the formula just proved to both sides of (4), obtaining

$$qy^{q-1}\,dy = pu^{p-1}\,du. \tag{5}$$

Substituting

$$y^{q-1} = \left(u^{\frac{p}{q}}\right)^{q-1} = u^{p-\frac{p}{q}}$$

and solving for dy we obtain

$$dy = \frac{p}{q}\,u^{\frac{p}{q}-1}\,du = nu^{n-1}\,du, \tag{6}$$

which proves the formula when n is a positive fraction.

Finally, if n is a negative rational number,

$$y = u^n = \frac{1}{u^{-n}}.$$

By formula V we then have

$$dy = \frac{u^{-n}d(1) - d(u^{-n})}{u^{-2n}}.$$

Since $-n$ is positive we can apply equation (6) just proved and so obtain

$$dy = \frac{nu^{-n-1}\,du}{u^{-2n}} = nu^{n-1}\,du.$$

Therefore whether n is an integer or fraction, positive or negative,

$$du^n = nu^{n-1}\,du.$$

If irrational powers are properly defined this formula is valid even when n is irrational. Proof of this, however, will be deferred to a later section (§88 and Problem 50, Chapter VII).

Example 1. $y = x^3 - 4x^2 - 9x + 7.$

Using formulas II, III, and VI in order we have

$$\begin{aligned} dy &= d(x^3) + d(-4x^2) + d(-9x) + d(7) \\ &= d(x^3) - 4\,d(x^2) - 9\,d(x) + d(7) \\ &= (3x^2 - 8x - 9)\,dx. \end{aligned}$$

Example 2. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}.$

This can be written

$$y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}.$$

Consequently, by II and VI

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}.$$

Example 3. $y = (x^2 - a^2)(x^2 - b^2).$

It is assumed that a and b are constants. Using IV with $u = x^2 - a^2$, $v = x^2 - b^2$,

$$\begin{aligned} \frac{dy}{dx} &= (x^2 - a^2)\frac{d}{dx}(x^2 - b^2) + (x^2 - b^2)\frac{d}{dx}(x^2 - a^2) \\ &= (x^2 - a^2)2x + (x^2 - b^2)2x \\ &= 4x^3 - 2(a^2 + b^2)x. \end{aligned}$$

Example 4. $y = \sqrt{x^2 - 1}$.
Using VI with $u = x^2 - 1$,

$$dy = \tfrac{1}{2}(x^2 - 1)^{-\frac{1}{2}}\, d(x^2 - 1)$$

$$= \frac{x\,dx}{\sqrt{x^2 - 1}}.$$

Example 5. $y = \dfrac{2x - 1}{2x + 3}$.

Using V with $u = 2x - 1$, $v = 2x + 3$,

$$dy = \frac{(2x + 3)\, d(2x - 1) - (2x - 1)\, d(2x + 3)}{(2x + 3)^2}$$

$$= \frac{(2x + 3)\, 2dx - (2x - 1)\, 2dx}{(2x + 3)^2}$$

$$= \frac{8\, dx}{(2x + 3)^2}.$$

22. Differentiation of Sine and Cosine.

VII. $d \sin u = \cos u\, du$.
VIII. $d \cos u = -\sin u\, du$.

To prove the first of these formulas let

$$y = \sin u. \tag{1}$$

Then

$$y + \Delta y = \sin (u + \Delta u) = \sin u \cos \Delta u + \cos u \sin \Delta u.$$

Subtracting the preceding equation,

$$\Delta y = \sin u\, (\cos \Delta u - 1) + \cos u \sin \Delta u,$$

whence

$$\frac{\Delta y}{\Delta u} = \cos u \frac{\sin \Delta u}{\Delta u} - \sin u \frac{1 - \cos \Delta u}{\Delta u}. \tag{2}$$

By equations (6) and (9) of §10 we have

$$\lim_{\Delta u \to 0} \frac{\sin \Delta u}{\Delta u} = 1, \quad \lim_{\Delta u \to 0} \frac{1 - \cos \Delta u}{\Delta u} = 0.$$

When Δu tends to zero equation (2) thus gives as limit

$$\frac{dy}{du} = \cos u,$$

and so

$$dy = \cos u\, du$$

which was to be proved.

To prove the second formula we note that

$$\cos u = \sin\left(\frac{\pi}{2} - u\right).$$

By the formula just proved we thus have

$$d\cos u = d\sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right) d\left(\frac{\pi}{2} - u\right) = -\sin u\, du.$$

Example 1. $y = 2\cos(3x + 4)$.
By formula VIII we have

$$dy = -2\sin(3x + 4)\, d\,(3x + 4) = -6\sin(3x + 4)\, dx.$$

Example 2. $y = \sin(3x^2)$.
Using formula VII

$$\frac{dy}{dx} = \cos(3x^2)\frac{d}{dx}(3x^2) = 6x\cos(3x^2).$$

23. Small Errors. If the error in measuring a quantity y is Δy, the ratio

$$\frac{\Delta y}{y}$$

is called the *relative error* and

$$100\frac{\Delta y}{y}$$

the *percentage error* in y.

When the relative error is small (as it usually is) a satisfactory approximation is obtained by replacing Δy by dy. Thus

$$\frac{dy}{y}$$

is usually taken as the relative error and

$$100\frac{dy}{y}$$

as the percentage error in y.

Example 1. The area of a rectangle is determined by measuring its sides x and y. If the measurement of x is 1% too large and that of y is $\frac{1}{2}$% too small, find the percentage error in the area.

From the expression

$$A = xy$$

for the area, by differentiation and division, we find

$$\frac{dA}{A} = \frac{dx}{x} + \frac{dy}{y}.$$

The relative error in A is thus the sum of the relative errors in x and y and the percentage errors, being 100 times as great, satisfy a similar equation. Thus

$$\text{percentage error in } A = 1 - \tfrac{1}{2} = \tfrac{1}{2}\%.$$

Example 2. The time of vibration of a pendulum of length l is determined by the equation

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Find the maximum error in the computed value of T due to errors of 1% in the measurements of l and g.

The expression for T can be written

$$T = 2\pi l^{\frac{1}{2}}g^{-\frac{1}{2}},$$

whence

$$dT = \pi l^{-\frac{1}{2}}g^{-\frac{1}{2}}\,dl - \pi l^{\frac{1}{2}}g^{-\frac{3}{2}}\,dg$$

and

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l} - \frac{1}{2}\frac{dg}{g}.$$

The relative error in T is thus one-half that in l minus one-half that in g. Since either measurement might be too large or too small the algebraic signs could be such that the errors add. Thus

$$\text{maximum error in } T = \tfrac{1}{2} + \tfrac{1}{2} = 1\%.$$

24. Derivatives of Higher Order.

The derivative of

$$y = f(x), \tag{1}$$

which we have written

$$\frac{dy}{dx} = f'(x), \tag{2}$$

is also a function of x. Its derivative with respect to x, written

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x), \tag{3}$$

is called the *second derivative* of y with respect to x. Similarly the third derivative is

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x), \tag{4}$$

etc.

For example, if

$$y = x^3,$$

we find

$$\frac{dy}{dx} = 3x^2,$$

$$\frac{d^2y}{dx^2} = 6x,$$

$$\frac{d^3y}{dx^3} = 6,$$

$$\frac{d^4y}{dx^4} = 0,$$

and all derivatives of higher order are zero.

From the notation

$$\frac{d^ny}{dx^n}$$

it might appear that the nth derivative is the ratio of two quantities d^ny and $(dx)^n$. These higher derivatives, however, do not have all the properties this fractional notation suggests.

In the product

$$\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2$$

one might think, for example, that $(dx)^2$ could be canceled in numerator and denominator, giving

$$\frac{d^2y}{dt^2}.$$

That this may not be true is shown by taking

$$y = x = t^2.$$

Then

$$\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 = 0$$

whereas

$$\frac{d^2y}{dt^2} = 2.$$

Consequently when n is greater than unity we do not consider

$$\frac{d^ny}{dx^n}$$

as a fraction but merely as a notation for the nth derivative of y with respect to x.

Since each derivative is the first derivative of the preceding one we can, however, write

$$\frac{d^ny}{dx^n} = \frac{d\left(\frac{d^{n-1}y}{dx^{n-1}}\right)}{dx}, \tag{5}$$

the right side being considered a fraction.

Thus if

$$x = t^2 + 1, \quad y = t^3 - 1$$

we have

$$\frac{dy}{dx} = \frac{d(t^3 - 1)}{d(t^2 + 1)} = \frac{3t^2\,dt}{2t\,dt} = \frac{3}{2}t,$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{3}{2}t\right)}{2t\,dt} = \frac{3}{4t}.$$

25. Implicit Functions. If x and y satisfy an equation

$$f(x, y) = 0,$$

not solved for y, then y is called an *implicit* function of x. To differentiate such a function we could solve for y and then differentiate. It may be simpler, however, to differentiate the given equation term by term and solve the resulting equation for the derivative.

Suppose, for example,

$$2xy + y^2 = 1. \tag{1}$$

Differentiating with respect to x,

$$2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0, \tag{2}$$

whence

$$\frac{dy}{dx} = -\frac{y}{x + y}. \tag{3}$$

To get the second derivative we differentiate again with respect to x, thus obtaining

$$\frac{d^2y}{dx^2} = \frac{y\left(1 + \frac{dy}{dx}\right) - \left(x + y\right)\frac{dy}{dx}}{(x + y)^2} = \frac{y - x\frac{dy}{dx}}{(x + y)^2}.$$

Replacing $\frac{dy}{dx}$ by its value from (3) and simplifying,

$$\frac{d^2y}{dx^2} = \frac{2xy + y^2}{(x + y)^3}.$$

Since x and y satisfy (1), this can be reduced further to the form

$$\frac{d^2y}{dx^2} = \frac{1}{(x + y)^3}.$$

By differentiating again with respect to x we could find the third derivative, etc.

26. Sign of the Derivative. At a point where $\frac{dy}{dx}$ is positive the curve

$$y = f(x)$$

slopes upward on the right, downward on the left. For sufficiently small changes, Δy then has the same sign as Δx and y increases as x increases. At a point where $\frac{dy}{dx}$ is negative the curve slopes downward on the right and y decreases as x increases. We thus have the theorem:

For a particular value of x, if $\frac{dy}{dx}$ is positive, y increases as x increases; if $\frac{dy}{dx}$ is negative, y decreases as x increases.

The curve will usually be divisible into arcs on which the slope is positive and others on which it is negative. For the curve in Figure 24, for example, the slope is positive on the left of A, negative between A and B, and positive on the right of B.

To determine for what values of x a given function $f(x)$ increases as x increases and for what values it decreases as x increases, we thus determine the values of x for which $f'(x)$ is positive and the values for which $f'(x)$ is negative.

If $f'(x)$ is a polynomial this is particularly easy. We merely factor the polynomial into a product of real factors and use the fact that a product is negative only if it contains an odd number of negative factors.

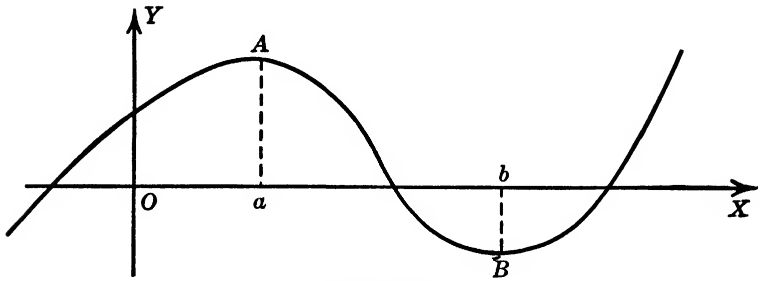

FIGURE 24.

Example. Determine the values of x for which

$$f(x) = x^4 - 4x^3 + 4x^2 + 1$$

increases as x increases and those for which this function decreases as x increases.

Differentiating and factoring,

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x - 1)(x - 2).$$

The values of x for which the right side is zero are $x = 0, 1, 2$. These divide all real numbers into four intervals.

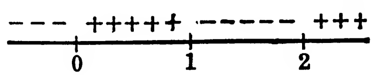

FIGURE 25.

1. If $x < 0$, all three factors in the product are negative and $f'(x)$ is negative.

2. If x is between 0 and 1, two factors are negative and the product positive.

3. If x is between 1 and 2, one factor is negative and the product negative.

4. If $x > 2$, all factors are positive and the product positive.

The algebraic signs for all real values of x are indicated graphically in Figure 25.

If

$$x < 0 \quad \text{or} \quad 1 < x < 2,$$

the derivative is negative and the function decreases as x increases. If

$$0 < x < 1 \quad \text{or} \quad x > 2,$$

the derivative is positive and the function increases as x increases.

The graph of the function is shown in Figure 26.

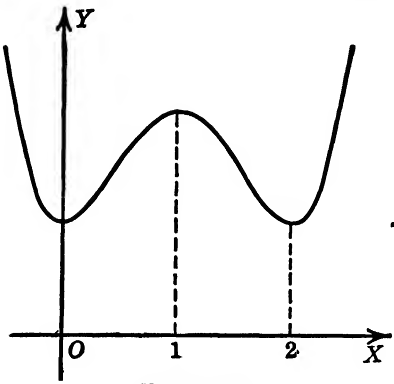

FIGURE 26.

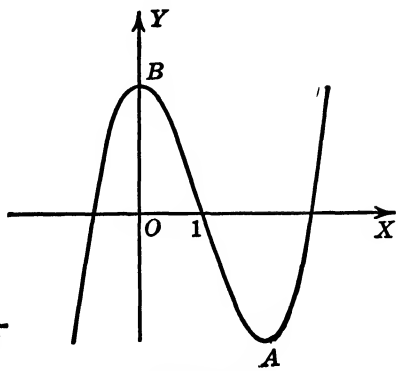

FIGURE 27.

27. Sign of the Second Derivative. The second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

is the derivative of the slope. At a point where the second derivative is positive the slope therefore increases as x increases and decreases as x decreases, and so the curve bends upward, or is concave upward, as at A in Figure 27. At a point where the second derivative is negative the slope decreases as x increases and increases as x decreases. The curve then bends downward, or is concave downward, as at B. That is,

A curve

$$y = f(x)$$

is concave upward at points where the second derivative

$$\frac{d^2y}{dx^2} = f''(x)$$

is positive, and concave downward at points where it is negative.

A point, on one side of which the curve is concave upward and the other concave downward, is called a *point of inflection*. On one side of such a point $\frac{d^2y}{dx^2}$ is positive and on the other side negative. To find points of inflection we thus determine positions at which the second derivative changes sign. At such a point the second derivative is usually zero, but it may be infinite or have a finite discontinuity.

Example. Find the values of x for which the curve

$$y = x^3 - 3x^2 + 2$$

is concave upward and the values for which it is concave downward. Also find the points of inflection.

In this case

$$\frac{d^2y}{dx^2} = 6(x - 1).$$

When $x > 1$ the second derivative is positive and the curve concave upward. When $x < 1$ the second derivative is negative and the curve concave downward. There is a point of inflection at $x = 1$, $y = 0$. The curve is shown in Figure 27.

28. Maxima and Minima. A function $f(x)$ is said to have a maximum at $x = a$ if when $x = a$ the function is greater than for any other value of x in the immediate neighborhood of a. It has a minimum at $x = a$ if when $x = a$ the function is less than for any other value of x sufficiently near a.

If we represent the function by y and plot the curve

$$y = f(x),$$

a maximum occurs at the top, a minimum at the bottom, of a wave. Thus in Figure 28 the function has a maximum at A and a minimum at B. It should be noted that a maximum is not necessarily the greatest and a minimum not necessarily the least value of the function.

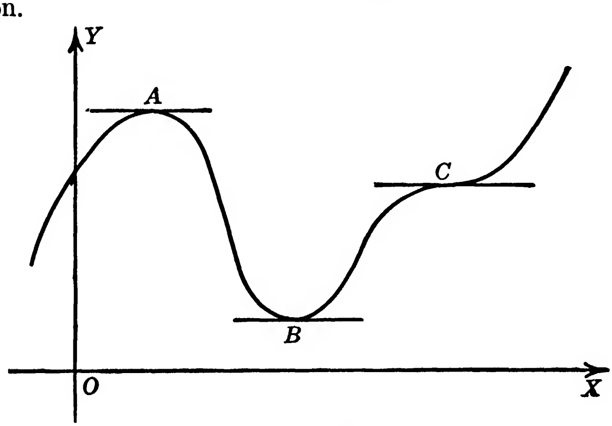

FIGURE 28.

There are several types of maxima and minima. At present we consider only the simplest, namely those represented graph-

ically by points such as A, B, Figure 28, the curve extending on both sides of the point and having a definite slope at the point. At a maximum or minimum of this type the slope is zero. For, if it were not zero the curve would rise on one side and fall on the other, and the function could not have either a maximum or a minimum at the point.

Hence in determining maxima and minima of a function $f(x)$ we first look for values of x such that

$$f'(x) = 0.$$

If $x = a$ is such a value, the function $f(x)$ may have a maximum at $x = a$, a minimum, or neither. Consideration of a diagram makes evident the following rule for distinguishing the three cases:

If $f'(x)$ changes from positive to negative as x increases through a, the function $f(x)$ has a maximum at $x = a$.

If $f'(x)$ changes from negative to positive as x increases through a, the function $f(x)$ has a minimum at $x = a$.

If $f'(x)$ has the same sign on both sides of $x = a$, the function $f(x)$ has neither a maximum nor a minimum at $x = a$.

Example 1. The sum of two numbers is 6. Find the maximum value of their product.

Let one of the numbers be x. The other is then $6 - x$. The value of x is to be found such that the product

$$y = x(6 - x) = 6x - x^2$$

is a maximum. The derivative

$$\frac{dy}{dx} = 6 - 2x = 2(3 - x)$$

is zero at $x = 3$. If x is less than 3 the derivative is positive and y decreases as x decreases. If x is greater than 3 the derivative is negative and y decreases as x increases. If x starts from 3 and varies steadily in either direction y steadily diminishes. The product therefore has the greatest possible value

$$y = x(6 - x) = 9$$

at $x = 3$ (Figure 29).

Example 2. Find the shape of the closed pint can which requires for its construction the least amount of tin.

Let the radius of base be r and the altitude h (Figure 30). The areas of bottom and top are each πr^2 and that of the side wall is $2\pi rh$. Hence the area of tin used is

$$A = 2\pi r^2 + 2\pi rh. \tag{1}$$

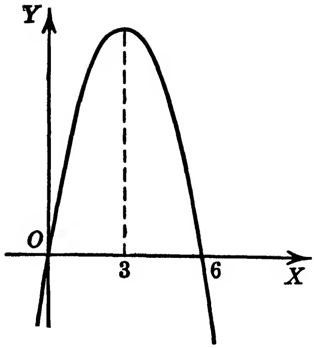

FIGURE 29.

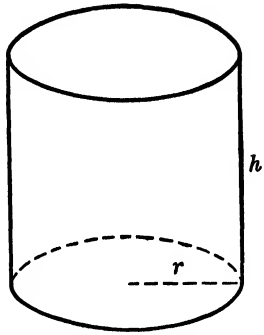

FIGURE 30.

Let v be the number of cubic inches in a pint. Then

$$v = \pi r^2 h. \tag{2}$$

Consequently

$$h = \frac{v}{\pi r^2}$$

and

$$A = 2\pi r^2 + \frac{2v}{r}. \tag{3}$$

Since π and v are constants

$$\frac{dA}{dr} = 4\pi r - \frac{2v}{r^2} = 2\left(\frac{2\pi r^3 - v}{r^2}\right). \tag{4}$$

This is zero when $2\pi r^3 = v$, that is, when

$$r = \sqrt[3]{\frac{v}{2\pi}}. \tag{5}$$

If a smaller value than this is substituted for r, $2\pi r^3$ will be less than v and $\dfrac{dA}{dr}$ negative. If a larger value is substituted, $2\pi r^3$ will be greater than v and $\dfrac{dA}{dr}$ positive. As r increases, $\dfrac{dA}{dr}$ changes from negative to positive and so the above value determines a minimum.

Another method of showing that (5) really determines a minimum is to observe that since the amount of tin cannot be zero there must be a least value. At this least value $\dfrac{dA}{dr}$ must be zero (see exceptions in §31). The derivative is zero for only one real value of r. That must then be the value which gives the minimum area.

Combining the equation

$$2\pi r^3 = v,$$

which determines the minimum, with the equation

$$v = \pi r^2 h$$

we see that $h = 2r$. The closed can requiring the least amount of tin thus has an altitude equal to the diameter of its base.

29. The Second Derivative Test. Suppose that

$$f'(a) = 0.$$

If $f''(a)$ is positive the curve $y = f(x)$ is concave upward (§27) and $x = a$ is a low point on the curve. If $f''(a)$ is negative, the curve is concave downward and $x = a$ is a high point. Therefore

If $f'(a) = 0$ and $f''(a) > 0$, the function $f(x)$ has a minimum at $x = a$. If $f'(a) = 0$ and $f''(a) < 0$, the function $f(x)$ has a maximum at $x = a$.

Example. Find the largest right circular cylinder that can be inscribed in a given right circular cone.

Let r be the radius and h the altitude of the cone, x the radius and y the altitude of the inscribed cylinder. From the similar triangles DEC and ABC (Figure 31)

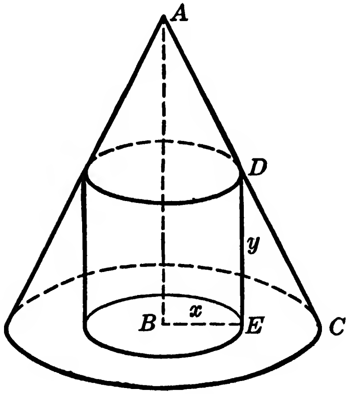

FIGURE 31.

$$\frac{DE}{EC} = \frac{AB}{BC},$$

that is

$$\frac{y}{r - x} = \frac{h}{r}, \quad y = \frac{h}{r}(r - x).$$

The volume of the cylinder is

$$v = \pi x^2 y = \pi \frac{h}{r}(rx^2 - x^3).$$

Hence

$$\frac{dv}{dx} = \frac{\pi h}{r}(2rx - 3x^2).$$

This is zero when $x = 0$ or $x = \frac{2}{3}r$. The value $x = 0$ obviously does not give the maximum. Hence

$$x = \tfrac{2}{3}r$$

is the only value to try. At this value the second derivative is

$$\frac{d^2v}{dx^2} = \frac{\pi h}{r}(2r - 6x) = -2\pi h.$$

Since the first derivative is zero and the second derivative negative the value of x found determines a maximum volume.

30. Method of Finding Maxima and Minima. The method we have used in solving maxima and minima problems involves the following steps:

(1) Decide what is to be a maximum or minimum. Let it be y.

(2) Express y in terms of a single variable. Let it be x.

It may be convenient to express y temporarily in terms of several variables. If the problem can be solved by our present methods, there will, however, be relations enough to eliminate all but one of these.

(3) Calculate $\frac{dy}{dx}$ and find for what values of x it is zero.

(4) It is usually easy to decide from the problem itself whether the corresponding values of y are maxima or minima. If not, determine the signs of $\frac{dy}{dx}$ when x is a little less and little greater than the values in question, and apply the criteria of §28, or find the second derivative and apply the criteria of §29.

31. Other Types of Maxima and Minima. The methods of §§28, 29 are sufficient for the determination of the maxima and minima of a function which occur at interior points of an interval within which the function and its derivative are continuous. In Figure 32 are shown some types that do not satisfy these conditions.

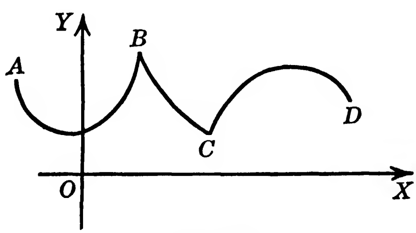

FIGURE 32.

At B the tangent is vertical and the derivative infinite. At C the slope has a finite discontinuity. At A and D the curve ends. This happens when values of the variable beyond a certain range are impossible. According to our definition the function has maxima at A, B and minima at C, D although the derivative is not zero at any of these points.

This diagram shows that in determining maxima and minima special attention must be given to values at which the derivative becomes infinite or discontinuous and values beyond which some condition becomes impossible.

Example 1. Find the maximum and minimum ordinates on the curve

$$y^3 = x^2.$$

In this case

$$y = x^{\frac{2}{3}}, \quad \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}.$$

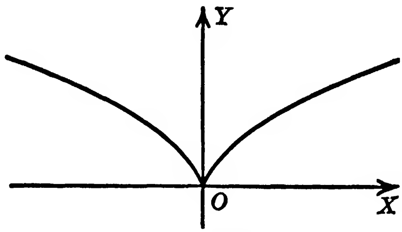

FIGURE 33.

No finite value of x makes the derivative zero but $x = 0$ makes it infinite. Since y is never negative the value $x = 0$, $y = 0$ is evidently a minimum (Figure 33).

Example 2. A man on one side of a river 1 mile wide wishes to reach a point on the opposite side 2 miles down the river. If he can row 6 miles per hour and walk 3, find the route he should take to make the trip in least time.

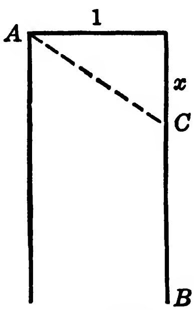

FIGURE 34.

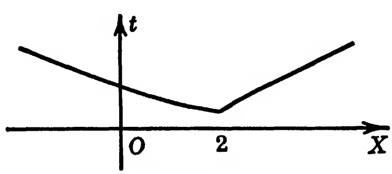

FIGURE 35.

Let A, Figure 34, be the starting point and B the destination. Suppose that he rows to C, x miles down the river. The total time is then

$$t = \tfrac{1}{6}\sqrt{x^2 + 1} + \tfrac{1}{3}(2 - x).$$

Equating the derivative to zero, we get

$$\frac{x}{6\sqrt{x^2+1}} - \frac{1}{3} = 0,$$

which reduces to

$$3x^2 + 4 = 0.$$

There is no real solution.

The trouble is that

$$\tfrac{1}{3}(2 - x)$$

is the time of walking only if C is above B. If C is below B, the time is

$$\tfrac{1}{3}(x - 2).$$

The complete value of t is thus

$$t = \tfrac{1}{6}\sqrt{x^2+1} \pm \tfrac{1}{3}(2 - x),$$

the sign being $+$ if $x < 2$ and $-$ if $x > 2$. The graph of the equation connecting x and t is shown in Figure 35. At $x = 2$ the derivative is discontinuous. Since he rows faster than he walks, the minimum obviously occurs when he rows all the way. That is, $x = 2$, $t = \frac{1}{6}\sqrt{5}$.

Example 3. Find the point $P(x, y)$ on the circle $x^2 + y^2 = 2x$ at shortest distance from $Q(-1, 0)$.

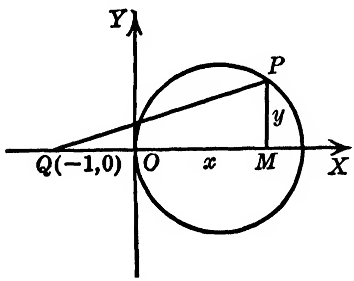

FIGURE 36.

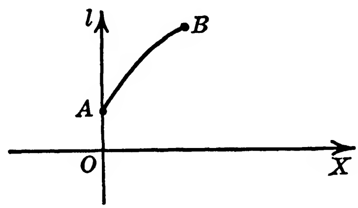

FIGURE 37.

The circle is shown in Figure 36. The triangle QMP has vertical side y and base $x + 1$. The distance QP is thus

$$l = \overline{QP} = \sqrt{(x+1)^2 + y^2}.$$

Replacing y^2 by its value $2x - x^2$ from the equation of the circle, this becomes

$$l = \sqrt{(x+1)^2 + 2x - x^2} = \sqrt{4x+1},$$

whence

$$\frac{dl}{dx} = \frac{2}{\sqrt{4x+1}}.$$

This does not become zero for any finite value of x. The explanation is that on the circle x lies between 0 and 2. The corresponding values of l are represented graphically by the curve AB in Figure 37. The minimum value of l occurs at A where $x = 0$. The shortest distance is therefore

$$l = \sqrt{4x + 1} = 1,$$

as is also obvious from Figure 36.

32. Velocity, Acceleration, Rate of Change. If t is the time and z is a function of t, the derivative

$$\frac{dz}{dt}$$

is called the *rate of change* of z. It is a function of the time which measures how fast z is changing at time t. When the rate of change is positive, z is increasing; when negative, z is decreasing.

Thus, if $s = f(t)$ is the distance a body travels in time t, the rate of change of s is its speed at time t. Whether the body moves one way or the other along its path, the distance traveled always increases and so the speed is always positive.

In general it is desirable to distinguish motion in one direction from motion in another. At present we consider only a particle moving along a straight line. Designate the position of the particle P at time t by means of its *displacement* from a fixed point O on the line. By this we mean a coördinate

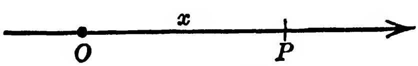

FIGURE 38.

$$x = OP$$

numerically equal to the distance of P from O but positive when P is on one side of O, negative when on the other. The rate of change of displacement

$$v = \frac{dx}{dt}$$

is called the *velocity* of P. It is numerically equal to the speed of P but positive or negative according as the motion is in the direction in which x increases or that in which x decreases.

The velocity of a moving particle is a function of the time. Its rate of change

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

is called the *acceleration* of the particle. When the acceleration is positive the velocity is algebraically increasing; when negative, the velocity is algebraically decreasing.

Displacement, velocity, and acceleration are usually considered directed quantities. When positive they are drawn in the direction in which x increases; when negative, that in which x decreases.

When the velocity and acceleration have the same direction, the change of velocity is in the direction of the velocity and so the speed is increasing. When the velocity and acceleration are oppositely directed the change of velocity is opposite to the velocity and so the speed is decreasing.

Example 1. The vertical height in feet of a ball thrown upward is

$$h = 100t - 16t^2$$

at the end of t seconds. Find its velocity and acceleration. Also find when it is rising, when falling, and when its speed is increasing and when decreasing.

The velocity and acceleration are

$$v = \frac{dh}{dt} = 100 - 32t,$$

$$a = \frac{dv}{dt} = -32.$$

The direction in which h increases is upward. The ball will be rising and the velocity directed upward while v is positive, that is, until

$$t = \tfrac{100}{32} = 3\tfrac{1}{8}.$$

It will be falling and the velocity directed downward after $t = 3\frac{1}{8}$. Since the acceleration is negative it is always downward. Before $t = 3\frac{1}{8}$ the velocity and acceleration are oppositely directed and so the speed decreases. After $t = 3\frac{1}{8}$ the velocity and acceleration have the same direction and the speed increases.

Example 2. A ship B sailing south 16 miles per hour is northwest of a ship A sailing east 10 miles per hour. At what rate are the ships approaching or separating?

Let x and y be the distances of the ships from the point where their paths cross. The distance between the ships is then

$$s = \sqrt{x^2 + y^2}.$$

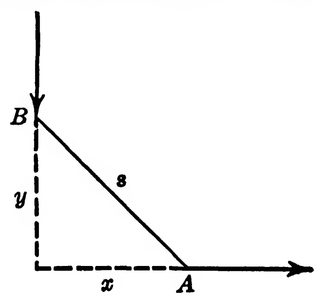

FIGURE 39.

The rate of change of s is the rate at which the ships are approaching or separating. By differentiation we find

$$\frac{ds}{dt} = \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

By hypothesis

$$\frac{dx}{dt} = 10, \quad \frac{dy}{dt} = -16,$$

$$\frac{x}{\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}} = \cos 45° = \frac{1}{\sqrt{2}}.$$

Therefore

$$\frac{ds}{dt} = \frac{10 - 16}{\sqrt{2}} = -3\sqrt{2} \text{ mi./hr.}$$

The negative sign shows that s is decreasing, that is, the ships are approaching.

33. Related Rates. In many problems the rates of change of certain variables are known and the rates of others are to be calculated. This is done by writing the equations connecting the variables and differentiating these with respect to the time. The resulting equations will contain both the known and the unknown rates. When values are substituted for the known rates, if the problem is determinate, there will be equations enough to solve for the unknown ones.

Example. The sides of a rectangle are $x = 5$ in., $y = 8$ in. If x is increasing 0.1 in./sec. and the area is decreasing 0.2 in.^{2}/sec., find the rate of change of y.

From the expression

$$A = xy$$

for area we obtain

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}.$$

Replacing x, y, $\dfrac{dx}{dt}$, and $\dfrac{dA}{dt}$ by the values stated, we have

$$-0.2 = 5\frac{dy}{dt} + 8(0.1),$$

whence

$$\frac{dy}{dt} = -0.2 \text{ in./sec.}$$

34. Mean Value Theorem. Let AB (Figure 40) be an arc of the curve $y = f(x)$. It is geometrically evident that there is some

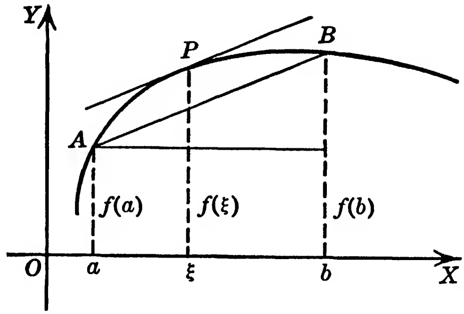

FIGURE 40.

point P on this arc at which the tangent is parallel to the chord AB. Equating the slope at P to the slope of the chord, we have

$$f'(\xi) = \frac{f(b) - f(a)}{b - a},$$

whence

$$f(b) - f(a) = (b - a)f'(\xi). \tag{1}$$

This is known as the mean value theorem. The following is a more precise statement:

Mean Value Theorem. *If $f(x)$ has a derivative for each value of x in the interval $a \leqq x \leqq b$, there is some value ξ between a and b such that*

$$f(b) - f(a) = (b - a)f'(\xi). \tag{2}$$

To prove this analytically consider the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{3}$$

which has been so constructed that

$$F(a) = F(b) = 0.$$

The result to be proved is equivalent to showing there is some value ξ between a and b such that

$$F'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0.$$

Now $F(x)$ is continuous in the interval $a \leqq x \leqq b$ and is equal to zero at $x = a$ and $x = b$. Either this function is zero for all

values of x between a and b, and then $F'(x) = 0$ for all such values, or there is some value ξ in the interval $a \leqq x \leqq b$ at which $F(x)$ is greatest or least. (Extreme Value Theorem, §14.) This extreme value cannot occur at an end of the interval, for there $F(x)$ is zero. Since $F(x)$ has a derivative at $x = \xi$, that derivative $F'(\xi)$ must be zero. For, if $F'(\xi)$ were different from zero, $F(x)$ would increase when x, starting from ξ, varies in one direction and decrease when x varies in the other direction and so could not have a greatest or least value at $x = \xi$.

In particular, if

$$f(a) = f(b) = 0,$$

the mean value theorem shows that there is some value ξ between a and b such that

$$f'(\xi) = 0.$$

This is called Rolle's theorem.

Rolle's Theorem. *If $f(a) = f(b) = 0$ and $f(x)$ has a derivative at each point of the interval $a \leqq x \leqq b$, there is some value ξ between a and b such that*

$$f'(\xi) = 0.$$

With two functions the following extension of the mean value theorem is sometimes useful:

If $f(x)$ and $\phi(x)$ have derivatives for all values of x in the interval $a \leqq x \leqq b$, and if $\phi'(x)$ does not vanish in that interval except possibly at $x = a$ or $x = b$, there is some value ξ between a and b such that

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(\xi)}{\phi'(\xi)}. \tag{4}$$

To prove this consider the function

$$F(x) = [f(x) - f(a)][\phi(b) - \phi(a)] - [\phi(x) - \phi(a)][f(b) - f(a)].$$

As this satisfies the conditions of Rolle's theorem there is a number ξ between a and b such that

$$F'(\xi) = f'(\xi)[\phi(b) - \phi(a)] - \phi'(\xi)[f(b) - f(a)] = 0.$$

Since $\phi'(x)$ does not vanish between a and b the mean value theorem shows that $\phi(b) - \phi(a)$ is not zero. From the last equation, (4) therefore follows by division.

Example. Find ξ in the mean value theorem if $a = 1$, $b = 4$, and

$$f(x) = 2x - \frac{3}{x}.$$

For this function

$$f'(x) = 2 + \frac{3}{x^2}$$

and equation (2) becomes

$$8\tfrac{1}{4} = \left(2 + \frac{3}{\xi^2}\right) 3,$$

whence

$$\xi = \pm 2.$$

Since ξ is between 1 and 4 the positive value must be taken, that is, $\xi = 2$.

35. Indeterminate Forms. When a particular value a is substituted for x, a function $F(x)$ may take one of the forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty.$$

These meaningless expressions are called *indeterminate.* When x tends to a, the function $F(x)$ may, however, tend to a definite limit. In that case we usually take that limit as defining $F(a)$.

Thus, when $x = 0$,

$$F(x) = \frac{\sin x}{x}$$

becomes $\frac{0}{0}$. When x tends to zero we have seen, however, that $F(x)$ tends to the limit 1, §10, (6). In a discussion involving this function we would therefore probably take $F(0)$ as 1.

At a point where a given function becomes indeterminate the limit can often be found by writing the function in an equivalent form that does not become indeterminate.

Thus, at $x = 1$,

$$\frac{x^2 - 1}{x - 1}$$

becomes $\frac{0}{0}$. For values of x different from 1 the function can be written

$$\frac{x^2 - 1}{x - 1} = x + 1,$$

and the right side obviously tends to the limit 2 when x tends to 1.

If the limit cannot be obtained in this simple way the following, known as *l'Hospital's rule*, may be useful:

If $\dfrac{f(x)}{\phi(x)}$ *assumes the form* $\dfrac{0}{0}$ *or* $\dfrac{\infty}{\infty}$ *when* $x = a$,

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}, \tag{1}$$

provided the limit on the right exists.

To give a complete proof of this rule it is necessary to consider several cases.

First suppose that a is finite and $f(a) = \phi(a) = 0$. If

$$\lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

exists there must be an interval of center a in which $f(x)$ and $\phi(x)$ are differentiable and $\phi'(x)$ does not vanish except possibly at $x = a$. For values of x in this interval, by §34, (4),

$$\frac{f(x)}{\phi(x)} = \frac{f(x) - f(a)}{\phi(x) - \phi(a)} = \frac{f'(\xi)}{\phi'(\xi)}, \tag{2}$$

ξ being between a and x. As x tends to a, ξ also tends to a and this equation gives as limit

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)},$$

which was to be proved.

Next suppose that a is finite but $f(x)$ and $\phi(x)$ become infinite at $x = a$. If

$$\lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

exists, there is an interval of center a in which $f(x)$ and $\phi(x)$ are differentiable and $f(x)$, $\phi(x)$, and $\phi'(x)$ do not vanish except possibly at $x = a$. Take c in this interval and x between c and a. By §34, (4), there is a value ξ between c and a such that

$$\frac{f(x) - f(c)}{\phi(x) - \phi(c)} = \frac{f'(\xi)}{\phi'(\xi)}, \tag{3}$$

whence

$$\frac{f(x)}{\phi(x)} = \frac{1 - \dfrac{\phi(c)}{\phi(x)}}{1 - \dfrac{f(c)}{f(x)}} \cdot \frac{f'(\xi)}{\phi'(\xi)}. \tag{4}$$

By taking c sufficiently close to a we can make the second factor on the right as nearly equal to its limit as we please, for all positions of ξ between c and a. Then leaving c fixed and taking x still closer to a the first factor can be made as nearly equal to 1 as we please. Thus there is an interval within which the difference

$$\frac{f(x)}{\phi(x)} - \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

is numerically less than any assigned number, whence (1) again follows.

Finally, by only slight modification of the above discussions we can show the results also valid when a is infinite.

In applying l'Hospital's rule, if $\frac{f'(x)}{\phi'(x)}$ also becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, numerator and denominator may again be replaced by their derivatives and this process repeated as many times as necessary. At any stage of the work it may be helpful to replace the fraction by an equivalent analytical form.

Other indeterminate forms, such as $0 \cdot \infty$, $\infty - \infty$, etc., can be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by analytical transformation.

Example 1. $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$.

When $x = 0$, numerator and denominator of this fraction are both zero. Replacing both by their derivatives, we obtain

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{2 \sin 2x}{2x} = \lim_{x \to 0} \frac{\sin 2x}{x}.$$

When $x = 0$ the fraction on the right becomes $\frac{0}{0}$ We therefore again replace numerator and denominator by their derivatives, thus finally obtaining

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2 \cos 2x}{1} = 2.$$

Example 2. $\lim\limits_{x \to a} \left[\dfrac{1}{x - a} - \dfrac{1}{\sin (x - a)}\right]$.

When $x = a$, the expression in brackets becomes $\infty - \infty$. Reducing to a common denominator we get

$$\frac{1}{x - a} - \frac{1}{\sin (x - a)} = \frac{\sin (x - a) - (x - a)}{(x - a) \sin (x - a)}$$

which becomes $\frac{0}{0}$ when $x = a$. Replacing numerator and denominator by their derivatives, we obtain

$$\frac{\cos (x - a) - 1}{(x - a) \cos (x - a) + \sin (x - a)}.$$

Since this is also $\frac{0}{0}$ when $x = a$, we again replace numerator and denominator by their derivatives, thus finally obtaining

$$\lim_{x \to a} \left[\frac{-\sin (x - a)}{2 \cos (x - a) - (x - a) \sin (x - a)}\right] = 0.$$

PROBLEMS

1. A particle moves the distance

$$s = 10t + 16t^2$$

in t seconds. Find its average speed during the first t seconds and its speed at time $t = 0$.

2. A body thrown upward with a speed of 100 feet per second rises to the height

$$h = 100t - 16t^2$$

in t seconds. Find its average speed between $t = 2$ and $t = 2.1$, and its speed at time $t = 2$.

3. A particle moves the distance $s = t^3$ feet in t seconds. Find its average speed between $t = 1$ and $t = 3$. Also find its speed at $t = 2$.

4. A particle moves the distance

$$s = t^3 - t^2 + 16t$$

in t seconds. Find its speed at time $t = 0$.

5. A body thrown downward with a speed of 50 feet per second falls the distance

$$h = 50t + 16t^2$$

feet in t seconds. Find its average speed during the first t seconds and its speed at the end of t seconds.

6. A particle, started with speed b and moving against an opposing force, travels the distance

$$s = bt - \tfrac{1}{2}kt^2$$

in t seconds, k being constant. Find when it comes to rest and its average speed during the period of motion.

7. Find the slope of the line through the points $P_1(1, 2)$ and $P_2(3, 6)$.

8. Find the slope of the line through the origin and the point $(1, -3)$.

9. Find the angle from the positive direction of the x-axis to the line joining $P_1(0, 2)$ and $P_2(3, 5)$.

10. Find the angle from the positive direction of the x-axis to the line joining $(3, -1)$ and $(-3, 5)$.

11. Find the slope of the curve $y = x^2 + x$ at the point $(1, 2)$.
12. Find the slope of the curve $y = 4x - x^3$ at the origin.
13. Find the slope of the curve $y = x^2 + x + 2$ at the point where $x = 1$.
14. Find the slope of the curve $y = x^3$ at the point where $x = x_1$.
15. Find the slope of the curve $y = x^n$ at $x = 0$, (*a*) if $n > 1$ (*b*) if $n = 1$.
16. Find the slope of the curve $y = \sin x$ at the origin [§10, (6)].
17. Find Δx when x changes from a to b.
18. If $y = x^3$, find Δy when x changes from 1 to 2.
19. Given $y = x^3$, find Δy if $x = 2$ and $\Delta x = -1$.

In each of the following problems show that the derivative has the value indicated:

20. $f(x) = 2x + 3, f'(x) = 2.$
21. $f(x) = x^2 - x, f'(x) = 2x - 1.$
22. $f(x) = x^4, f'(x) = 4x^3.$
23. $f(x) = \dfrac{1}{x+1}, f'(x) = -\dfrac{1}{(x+1)^2}.$
24. $f(x) = \dfrac{1}{x^2}, f'(x) = -\dfrac{2}{x^3}.$
25. $f(x) = \sqrt{x-1}, f'(x) = \dfrac{1}{2\sqrt{x-1}}.$

In each of the following problems find dy, assuming that x is the independent variable:

26. $y = x^3 - x.$
27. $y = x^2 + 3x + 2.$
28. $y = \sqrt{x}.$
29. $y = \dfrac{x}{x-1}.$

30. If $y = x^2$, $x = t^3$, and t is the independent variable, find dx and dy, and show that they satisfy the equation

$$dy = 2x\,dx.$$

31. If $x = \dfrac{1}{t}$, $y = t^2$, and t is the independent variable, find dy and dx, and show that their ratio is the derivative of y with respect to x.

32. Suppose that t is the independent variable and s the number of miles an automobile travels in t hours. Taking $\Delta t = 1$, make a statement concerning ds or Δs in each of the following cases:
(*a*) At time t the automobile is traveling 60 miles per hour.
(*b*) During the following hour the automobile travels 60 miles.

33. Let P be the population of a country at time t measured in years. Taking t as independent variable and $\Delta t = 1$:
(*a*) Find dP if the population is 1,000,000 and is increasing 1% per year.
(*b*) Find ΔP if the population is 1,000,000 and increases 1% in 1 year.

34. A coat of paint of thickness t is put on the surface of a sphere of radius r. Make a common-sense formula for the volume of paint used. If v is the vol-

ume of the sphere and r is the independent variable, determine dv when $\Delta r = t$.

35. A coat of paint of thickness t is put on a cube of side x and volume V. Determine approximately the volume of paint used. If x is the independent variable and $x + \Delta x$ is the side after painting, find Δx and dV.

36. Let V be the volume of water in a reservoir, A the area of its surface, and h the depth of water over a fixed level, all measured in feet. Determine approximately the increase in supply when the water level rises 1 inch. Taking h as independent variable, determine dV when Δh is 1 inch.

37. When a circular plate is heated its radius increases from 10 in. to 10.1 in. Determine approximately the increase in area.

38. If x is the independent variable and $f(x) = \sqrt{x}$, find $\Delta f(x)$ and $df(x)$: (a) when x changes from 25 to 26; (b) when x changes from 25 to 25.01. In each case express the difference $\Delta f(x) - df(x)$ as a fraction of Δx.

39. If x is the independent variable and $f(x) = \dfrac{1}{x}$, find $\Delta f(x) - df(x)$. (a) when x changes from 1 to 1.1; (b) when x changes from 1 to 1.001. In each case express the difference as a fraction of Δx.

Find $\dfrac{dy}{dx}$ in each of the following exercises:

40. $y = x^4 - 4x^3 + 2x^2 + 4x - 6.$

41. $y = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + 1.$

42. $y = 4(x^3 - 3x^2 + 3x).$

43. $y = \frac{1}{3}(x^3 + 3x^2 + 4).$

44. $y = x^{\frac{1}{2}} + x^{\frac{3}{2}}.$

45. $y = \sqrt{2x} + \dfrac{1}{\sqrt{2x}}.$

46. $y = 2\sqrt{x^3} + \dfrac{6}{\sqrt{x}}.$

47. $y = (2x - 1)^3.$

48. $y = (x^2 + x + 1)^4.$

49. $y = \sqrt{x^2 - 2x + 3}.$

50. $y = (x - 1)^3(3x - 1).$

51. $y = (x + a)(x^2 - ax + a^2).$

52. $y = (x + 1)^2(2x - 3)^2.$

53. $y = (x - 2)\sqrt{x + 1}.$

54. $y = x\sqrt{2 - x^2}.$

55. $y = (3x^2 - 2a^2)(x^2 + a^2)^{\frac{3}{2}}.$

56. $y = \dfrac{3x + 4}{2x + 3}.$

57. $y = \dfrac{x^2}{1 - x^2}.$

58. $y = \dfrac{\sqrt{4x^2 - 1}}{x}.$

59. $y = \dfrac{x}{\sqrt{a^2 - x^2}}.$

60. $y = (x^3 - 2)^{\frac{4}{3}}.$

61. $y = \dfrac{x + 1}{\sqrt{x^2 + 2x + 2}}.$

62. $y = \dfrac{2x^2 - 1}{3x^3}\sqrt{x^2 + 1}.$

63. $y = \dfrac{x^2}{(2x^6 + 1)^{\frac{1}{3}}}.$

64. $y = \dfrac{(3x - 1)^4}{x^4}.$

65. $y = 3 \sin (4x).$

66. $y = 2 \cos \left(x + \dfrac{\pi}{3}\right).$

67. $y = x - \sin x.$

68. $y = x \sin x + \cos x.$

69. $y = \sin^2 (2x + 3).$

70. By differentiating both sides of the equation

$$\sin 2x = 2 \sin x \cos x$$

obtain a formula for $\cos 2x$.

71. Leaving y constant and differentiating both sides of the equation

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

with respect to x, obtain a formula for $\cos (x + y)$.

72. Find the angle between the x-axis and the tangent to the curve

$$y = \frac{x}{1 + x^2}$$

at the origin.

73. Find the points on the curve

$$y = (x - 1)^2(x + 2)^2$$

where the tangent is parallel to the x-axis.

74. If $x = \cos \theta$, $y = \sin \theta$, find dx and dy, and show that their ratio is the derivative of y with respect to x.

75. By differentiating both sides of the equation

$$\cos^2 \theta + \sin^2 \theta = 1$$

and equating derivatives, do we get a correct result?

76. By differentiating both sides of the equation

$$x^2 + 2x = 1$$

and equating derivatives, do we get a correct result?

77. The area of a circle is calculated on the assumption that the radius is 8 inches. If the radius is actually 7.8 inches, determine the relative error and the percentage error in the calculated area.

78. Determine the allowable percentage error in the diameter of a sphere if the calculated volume is to be correct to 0.3%.

79. The time of vibration of a simple pendulum of length l is

$$T = 2\pi \sqrt{\frac{l}{g}}.$$

If a clock loses 1 second per day, determine approximately the error in the length of its pendulum.

80. The volume of a cylinder is determined from its radius and altitude. If the measurements of radius and altitude are correct to within 1%, determine the maximum error in the calculated volume.

81. The radius of a spherical ball is calculated from the weight of the ball and the density of its material. If an error of 0.5% is made in weighing the ball and 1% in determining its density, what is the maximum error that may result in the determination of the radius?

Find $\dfrac{d^2y}{dx^2}$ in each of the following problems:

82. $y = \sqrt{2x + 3}$.

83. $y = \sin 2x$.

84. $y = (2 - x^2)^{\frac{5}{2}}$.

85. $xy + 3y - x + 1 = 0$.

86. $y^2 = 2ax$.

87. $x^2 + y^2 = 5$.

88. $y^2 - x^2 = 1$.

89. $x^2 + xy + y^2 = 5$.

90. $x^2 + 2xy - y^2 = 0$.

91. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$.

92. $x = \cos\theta,\ y = \sin\theta$.

93. $x = t + \dfrac{1}{t},\ y = t - \dfrac{1}{t}$.

94. Find the derivative of

$$t^2\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x$$

with respect to t.

95. Given $y = x^3$, $x = t^2$, show that

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dt}\right)^3$$

is not equal to

$$\frac{d^3y}{dt^3}.$$

Find the values of x for which the following functions increase as x increases and the values for which they decrease. Also find the values at which the curves are concave upward and those at which they are concave downward:

96. $y = x^2 - 4x + 4$.

97. $y = x^3 - 3x$.

98. $y = x^3 + 3x^2 + 4$.

99. $y = x - \dfrac{1}{x}$.

100. $y = x^4 - 4x^3 + 1$.

101. $y = x^4 + 4x + 3$.

102. Show that

$$x^3 - 3x^2 + 3x + 2$$

has no maximum or minimum.

103. Show that

$$x + \frac{1}{x}$$

has a maximum and a minimum but that the maximum is less than the minimum.

104. A piece of wire 40 inches long is bent into a rectangle. Find the maximum area of the rectangle.

105. Find the minimum perimeter of a rectangle of area 100 square inches.

106. A box with a square base and open at the top is to be made from 300 square inches of material. If no allowance is made for thickness of material or waste in construction, find the dimensions of the largest box that can be made.

107. A box is to be made from a piece of cardboard 6 inches square by cutting equal squares from the corners and turning up the sides. Find the dimensions of the largest box that can be made.

108. A tin can open at the top is to have a given capacity. Find the ratio of height to diameter if the amount of tin required is the minimum.

109. Find the volume of the largest right circular cone that can be inscribed in a sphere of volume 27 cubic inches.

110. Find the altitude of the largest cone that can be generated by rotating a right triangle of hypotenuse 2 feet about one of its sides.

111. Among all circular sectors with a given perimeter find the one which has the greatest area.

112. The same distance was measured four times, the results being a_1, a_2, a_3, a_4. By the method of least squares the most probable value for the correct distance is that which makes the sum of the squares of the four errors least. What is that value?

113. A gutter of trapezoidal section is made by joining three flat strips each 4 inches wide, the middle one being horizontal and the other two inclined at the same angle. How wide should the gutter be at the top to have maximum capacity?

114. To reduce the friction of a liquid against the walls of a channel, the channel is sometimes so designed that the area of wetted surface is as small as possible. Determine the ratio of width to depth for the best open rectangular channel with given cross-section area.

115. A circular filter paper of radius 3 inches is to be folded into a conical filter. Find the radius of the base of the filter if it has maximum capacity.

116. What are the most economical dimensions of an open cylindrical water tank if the cost of the sides per square foot is twice that of the bottom?

117. If the top and bottom margins of a printed page are each of width a, the side margins of width b, and the printed matter covers an area c, what should be the dimensions of the page to use the least paper?

118. Assuming that the intensity of light is inversely proportional to the square of the distance from the source, find the point on the line joining two sources, one of which is 8 times as intense as the other, at which the illumination is least.

119. A ship B is 75 miles due east of a ship A. If B sails west 12 miles per hour and A south 9 miles per hour, find when the ships will be closest together.

120. A fence a feet high runs parallel to, and b feet from, a wall. Find the shortest ladder that will reach from the ground over the fence to the wall.

121. A corridor of width b runs at right angles to a passageway of width a. What is the longest beam that can be moved in a horizontal plane along the passageway into the corridor? Show that the value obtained is a minimum. Explain.

122. A and C are points on the same side of a plane mirror. A ray of light passes from A to C by way of a point B on the mirror. Show that the length of path will be least when the lines BA, BC make equal angles with the perpendicular to the mirror.

123. Let the velocity of light in air be V_1 and in water V_2. The path of a ray of light from a point A in the air to a point C below the surface is bent at B where it enters the water. If θ_1 and θ_2 are the angles that AB and BC

make with the perpendicular to the surface, show that the time required for light to pass from A to C will be least if B is so placed that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{V_1}{V_2}.$$

124. Assuming that the cost per hour of propelling a steamer is proportional to the cube of its speed through the water, find the speed at which it should be run against a current of 5 miles per hour to make a given trip at least cost.

125. In the preceding problem find the most economical speed if the steamer moves with the current.

126. A wire of length L is cut into two pieces, one of which is bent to form a circle, the other a square. Find the lengths of the two pieces when the sum of the areas of the square and circle is greatest.

127. Find the point (x, y) on the curve $y^2 = 2(x - 1)$ at shortest distance from the origin.

128. Find the point on the x-axis the sum of whose distances from $(0, 1)$ and $(1, 0)$ is least.

129. Find the minimum value of the function

$$y = 1 + (x - 1)^{\frac{2}{3}}.$$

130. Find the maximum value of the function

$$y = x - 2\,|\,x - 1\,|.$$

131. A log is 12 feet long and has diameters 4 feet and 5 feet at its ends. Find the volume of the largest cylinder that can be cut from the log.

132. A particle moves along a straight line,

$$s = 32t - t^4$$

being its displacement from a fixed point of the line at time t. Find its acceleration at the point where its velocity is zero.

133. At time t the displacement of a particle from a fixed point of its line of motion is

$$s = t^3 - 6t^2 + 9t.$$

(*a*) During what interval is its velocity negative? (*b*) During what interval does its velocity decrease?

In each of the following problems a particle moves along a line, x being its displacement from a fixed point of the line at time t. At the instant indicated determine, (*a*) whether the particle is moving toward the fixed point or away, (*b*) whether it is accelerated toward the fixed point or away, (*c*) whether the speed is increasing or decreasing:

134. $x = \sin 3t, \quad t = \dfrac{\pi}{4}.$

135. $x = 3 + 5t - t^2, \quad t = 2.$

136. $x = t^2 - 4t - 1, \quad t = 0.$

137. $x = 1 - \cos t, \quad t = \dfrac{\pi}{3}.$

138. $x = t^3 + 6t, \quad t = -1.$

139. $x = t^4 - 8t^2, \quad t = 1.$

140. At time t the displacement of a particle from a fixed point of its line of motion is

$$s = \sqrt{k^2t^2 + C},$$

k and C being constants. Show that the velocity of the particle approaches k as limit as t tends to infinity and that its acceleration is inversely proportional to the cube of s.

141. A body, starting from rest, has the velocity

$$v = 8\sqrt{h} \text{ ft./sec.}$$

when it has fallen h feet. By differentiating with respect to the time, and eliminating

$$\frac{dh}{dt} = v$$

determine its acceleration.

142. A particle, starting from rest at an indefinitely great distance and falling under gravity toward the earth, has the velocity v determined by the equation

$$v^2 = \frac{k}{r}$$

when at distance r from the center of the earth. By differentiating with respect to the time find its acceleration, determining the constant k from the known acceleration g at the surface of the earth.

143. A particle moves along a straight line, x being its displacement from a fixed point of the line, v its velocity, and a its acceleration. Show that

$$v\frac{dv}{dx} = a.$$

144. A point moves along the parabola

$$x^2 = 2py$$

in such a way that its projection on the x-axis has constant velocity. Show that its projection on the y-axis has constant acceleration.

145. A ladder 50 feet long rests against a vertical wall. If the foot of the ladder is pulled away from the wall at the rate of 3 feet per second, how fast is the top moving when the foot is 30 feet from the wall?

146. A man 6 feet tall walks at the rate of 5 feet per second directly away from a lamp 15 feet above the ground. Find the rate at which the end of his shadow is moving and the rate at which his shadow is growing when he is 20 feet away from the lamp post.

147. A ball falls vertically

$$s = 16t^2$$

feet in t seconds from a point 64 feet above the ground. Find the speed of its shadow on the ground just before the ball strikes if the sun's rays make an angle of 60° with the ground.

148. A kite is 300 feet high, and 500 feet of cord are out. Assuming the cord to stretch in a straight line, if the kite moves horizontally 10 feet per

second directly away from the person flying it, how fast is the cord being paid out?

149. The side of a square is increasing 10 feet per minute and its area 100 square feet per minute. Find the side of the square.

150. A basin has the form of an inverted cone of altitude 10 inches and diameter of base 10 inches. If water runs in at the rate of 5 cubic inches per second, how fast is the water level rising when the depth is 6 inches?

151. At what rate is the wetted surface increasing in the preceding problem?

152. A reservoir covers 100 acres. At what rate must water be pumped in to raise the level 1 foot per day?

In each of the following, find the value ξ in the mean value theorem:

153. $f(x) = x^2 - 4x + 3, \quad a = 1, b = 4.$

154. $f(x) = x^2 + 4x, \quad a = -3, b = 5.$

155. $f(x) = x^3 + 7x^2 - 36, \quad a = -3, b = 2.$

156. $f(x) = x^{\frac{3}{2}}, \quad a = 0, b = 9.$

157. $f(x) = x + \dfrac{1}{x}, \quad a = 1, b = 9.$

158. $f(x) = \dfrac{1}{x+1}, \quad a = 0, b = 8.$

159. If $a = -1, b = 1,$

$$f(x) = x^{\frac{2}{3}},$$

show that there is no value ξ that satisfies the mean value theorem. Explain.

160. Given $f(x) = x^3$, $\phi(x) = x^2$, $a = -3$, $b = 2$, show that there is no value ξ between a and b that satisfies equation (4), §34. Explain.

Determine the following limits:

161. $\displaystyle\lim_{x \to 1} \frac{x^{10} - 1}{x^9 - 1}.$

162. $\displaystyle\lim_{x \to a} \frac{x^{\frac{5}{2}} - a^{\frac{5}{2}}}{x - a}.$

163. $\displaystyle\lim_{x \to 0} \frac{x - \sin x}{x^3}.$

164. $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\pi - 2x}.$

165. $\displaystyle\lim_{x \to 0} \frac{2 - x - 2\sqrt{1 - x}}{x^2}.$

166. $\displaystyle\lim_{x \to 1} \frac{(2 - x)^{\frac{2}{3}} - x^{\frac{2}{3}}}{x - 1}.$

167. $\displaystyle\lim_{x \to \frac{\pi}{2}} \left[\sec x + \frac{1}{x - \frac{\pi}{2}}\right].$

168. $\displaystyle\lim_{x \to 0} \left[\frac{\cos x}{\sin x} - \frac{1}{x}\right].$

169. $\displaystyle\lim_{x \to 0} \frac{\sin 2x - 2 \sin x}{x^3}.$

170. $\displaystyle\lim_{x \to \pi} \frac{2 \sin x + \sin 2x}{\cos x + \cos 2x}.$

CHAPTER III

INTEGRATION AND SUMMATION

36. Integral. In many problems the derivative or differential of a function is known and it is necessary to find the function. Thus, if the speed of a moving particle at time t is

$$\frac{ds}{dt} = f(t),$$

the distance it moves in time t is a quantity s with differential

$$ds = f(t)\,dt.$$

To find the distance we must then find a function of t with differential equal to $f(t)\,dt$.

The process of finding a function with given differential is called *integration*. If

$$dF(x) = f(x)\,dx$$

then $F(x)$ is called an integral of $f(x)\,dx$, and this is indicated by the notation

$$F(x) = \int f(x)\,dx.$$

By differentiation we pass from a function to its differential. By integration we pass from the differential to the function. Integration is thus the inverse of differentiation.

For example, since $d(x^2) = 2x\,dx$,

$$\int 2x\,dx = x^2.$$

Similarly,

$$\int \cos x\,dx = \sin x, \qquad \int \sin x\,dx = -\cos x.$$

The test of integration is to differentiate the answer. If the integration is correct, the differential of the answer must equal the expression to be integrated.

It should be noted that we always integrate a differential and not a derivative. There are several reasons for this. A very important one is the fact that the differential of a quantity is independent of the variable in terms of which it is expressed. If the same quantity y is expressible in terms of two different variables

$$y = f_1(x), \quad y = f_2(t),$$

its differential has equal values whether the one or the other of these expressions is used. But the derivatives of y with respect to x and t are not in general equal. If integration were defined as the process of finding a function with given derivative, there would be a different integral for every variable.

37. Constant of Integration. In solving a problem by integration we need to know whether the given differential has more than one integral; for, if there are several different integrals, we must choose the one that represents the answer.

As a matter of fact each differential has an infinite number of integrals but they are related in a rather simple way. If

$$dF(x) = f(x)\,dx$$

and C is any constant,

$$d[F(x) + C] = d\,F(x) = f(x)\,dx.$$

Thus, if $F(x)$ is one integral of a given differential,

$$F(x) + C$$

is another.

Conversely, *if two functions have equal differentials at all points of an interval, their difference is constant in that interval.*

To prove this suppose that

$$dF_1(x) = dF_2(x) = f(x)\,dx$$

at all points of the interval $a \leqq x \leqq b$. Let

$$\phi(x) = F_2(x) - F_1(x).$$

By hypothesis

$$\phi'(x) = \frac{dF_2(x)}{dx} - \frac{dF_1(x)}{dx} = 0$$

at all points of the interval $a \leqq x \leqq b$. For any value of x in that interval the mean value theorem [§34, (1)] then gives

$$\phi(x) - \phi(a) = (x - a)\,\phi'(x_1) = 0.$$

Thus

$$\phi(x) = \phi(a)$$

and so the difference

$$F_2(x) - F_1(x) = \phi(x)$$

has the same value $\phi(a)$ at all points of the interval.

If then $F(x)$ is one integral of a given differential $f(x)\,dx$, any other has the form

$$\int f(x)\,dx = F(x) + C.$$

Any constant value can be assigned to C. It is therefore called an *arbitrary* constant. In the statement of a definite problem there is usually some information by which this constant can be determined.

38. Integration Formulas. Integration is usually performed by means of formulas of which the following will be sufficient for the applications made in this chapter. In these a, n, c are constants and u, v functions of a single variable.

I. $\displaystyle\int (du + dv) = \int du + \int dv.$

II. $\displaystyle\int a\,du = a\int du.$

III. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$ if n is not -1.

IV. $\displaystyle\int \sin u\,du = -\cos u + C.$

V. $\displaystyle\int \cos u\,du = \sin u + C.$

Any one of these formulas can be proved by showing that the differential of the right member is equal to the expression under the integral sign.

Formula I states that the integral of a sum is equal to the sum of the integrals of its parts. Integration like differentiation is thus a distributive operation (§21).

Formula II states that a constant factor may be transferred from one side of the integral sign to the other without changing the result. Integration and multiplication by a constant are thus

commutative operations. It should be noted that a variable cannot be transferred in this way. Thus

$$\int x\,dx$$

is not equal to

$$x\int dx.$$

In applying formulas III, IV, and V it is important that du be the differential of the quantity called u. Thus we cannot evaluate

$$\int \cos 2x\,dx$$

by immediate application of V, for dx is not the differential of $2x$. We can, however, write

$$dx = \tfrac{1}{2}\,d(2x)$$

and so obtain

$$\int \cos 2x\,dx = \tfrac{1}{2}\int \cos 2x\,d(2x) = \tfrac{1}{2}\sin 2x + C.$$

Example 1. $\int (x-1)(x+2)\,dx.$

Expanding and integrating term by term, we obtain

$$\int (x-1)(x+2)\,dx = \int (x^2+x-2)\,dx = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 - 2x + C.$$

Example 2. $\int \sqrt{2x+1}\,dx.$

Taking $2x + 1 = u$, we have

$$dx = \tfrac{1}{2}\,du$$

and

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{\tfrac{1}{2}u^{\frac{3}{2}}}{\tfrac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

Example 3. $\int \sin (ax+b)\,dx.$

Taking $ax + b = u$, we have

$$dx = \frac{1}{a}\,du$$

and

$$\int \sin (ax+b)\,dx = \frac{1}{a}\int \sin u\,du = -\frac{1}{a}\cos (ax+b) + C.$$

39. Variable with Given Rate of Change. If the rate of change of a variable z is a known function of the time

$$\frac{dz}{dt} = f(t),$$

integration gives

$$z = \int f(t)\, dt + C$$

as its value at time t.

The constant C can be determined if we know the value of z at some particular time. Thus, if z has the value z_0 at time t_0, we find C by substituting

$$z = z_0, \quad t = t_0$$

after integration.

Example 1. From a point 60 feet above the ground a ball is thrown upward with a velocity of 100 feet per second. If it is subject only to the acceleration of gravity, find its height t seconds after the beginning of motion.

Let y be the height (considered positive upward), and

$$v = \frac{dy}{dt}$$

the velocity of the ball at time t. Its acceleration is then

$$\frac{dv}{dt} = -g,$$

the negative sign being used because the acceleration is opposite to the direction in which y is measured. By integration we find

$$v = -gt + C_1,$$

C_1 being constant. Substituting the initial values

$$v = 100, \quad t = 0,$$

we obtain

$$100 = C_1,$$

whence

$$\frac{dy}{dt} = v = -gt + 100.$$

A second integration gives

$$y = -\tfrac{1}{2}gt^2 + 100t + C_2.$$

Substituting

$$y = 60, \quad t = 0$$

we find the constant is equal to 60 and so

$$y = -\tfrac{1}{2}gt^2 + 100t + 60$$

at time t.

Example 2. Suppose that, through condensation upon its surface, a spherical drop of liquid grows at a rate proportional to its area of surface. If it starts with radius r_o, find its radius at time t.

Let r be the radius, V the volume, and S the surface area of the drop at time t. By hypothesis

$$\frac{dV}{dt} = kS,$$

where k is constant. Substituting

$$V = \tfrac{4}{3}\pi r^3, \quad S = 4\pi r^2,$$

we obtain

$$4\pi r^2 \frac{dr}{dt} = k\cdot 4\pi r^2,$$

whence

$$\frac{dr}{dt} = k.$$

Integration gives

$$r = kt + C,$$

C being constant. Substituting $r = r_o$, $t = 0$, we have

$$r_o = C,$$

and so

$$r = kt + r_o$$

is the value of the radius at time t.

40. Curve with Given Slope. If the slope of a curve is a given function of x,

$$\frac{dy}{dx} = f(x),$$

integration gives

$$y = \int f(x)\,dx + C$$

as the equation of the curve.

Since the constant may have any value there are infinitely many

curves having the given slope. These are obtained from one,

$$y = \int f(x)\,dx,$$

by moving it through various distances C in a vertical direction. If the curve is required to pass through a given point P, the value

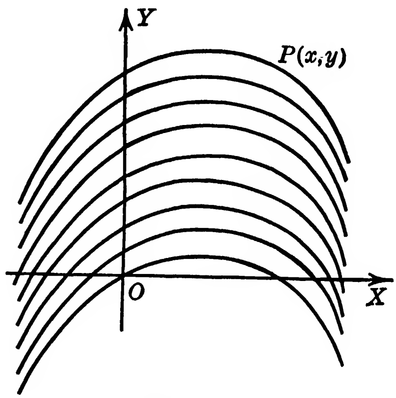

FIGURE 41.

of C can be found by substituting the coördinates of P after integration.

Example 1. Find the curve passing through (1, 2) with slope equal to $2x$.

In this case

$$\frac{dy}{dx} = 2x.$$

Hence

$$y = \int 2x\,dx = x^2 + C.$$

Since the curve passes through (1, 2) the values $x = 1$, $y = 2$ satisfy the equation; that is,

$$2 = 1 + C.$$

Consequently $C = 1$, and

$$y = x^2 + 1$$

is the required curve.

Example 2. On a certain curve

$$\frac{d^2y}{dx^2} = 6x.$$

If the curve passes through $(-2, 1)$ and has at that point the slope 4, find its equation.

By integration we get

$$\frac{dy}{dx} = \int 6x\,dx = 3x^2 + C_1.$$

At $(-2, 1)$, $x = -2$ and $\dfrac{dy}{dx} = 4$. Hence

$$4 = 12 + C_1,$$

or $C_1 = -8$. Thus

$$\frac{dy}{dx} = 3x^2 - 8$$

and so

$$y = x^3 - 8x + C_2.$$

Since the curve passes through $(-2, 1)$, $C_2 = -7$, and

$$y = x^3 - 8x - 7$$

is the equation of the curve.

41. Area as Limit. The methods of elementary geometry determine directly only the areas of regions with straight-line boundaries. To define the area of a plane region with curved boundary it is customary to consider the region as limit of a set of regions with rectilinear boundaries.

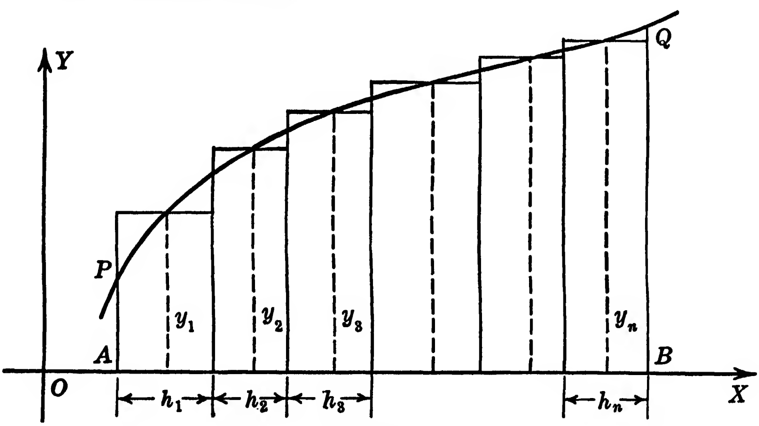

FIGURE 42.

Thus to define the area $ABQP$ (Figure 42) bounded by the x-axis, a curve

$$y = f(x)$$

above the x-axis, and ordinates at A and B, divide AB into equal

or unequal parts $h_1, h_2, \cdots, h_n$, and on each part h_i as base construct a rectangle of altitude equal to the ordinate y_i at some point of the arc above this part. The set of rectangles thus constructed forms a polygonal region which is taken as an approximation for the region bounded by the curve, and the sum of the areas of those rectangles

$$S_n = h_1y_1 + h_2y_2 + \cdots + h_ny_n$$

is taken as an approximation for the area between the x-axis and the curve.

Now suppose that the number n of parts into which the base AB is divided is made greater and greater but in such a way that the lengths $h_1, h_2, \cdots h_n$ of all the parts tend to zero as n tends to infinity. If the sum of the areas of the rectangles tends to a limit (as it always does in cases in which we are interested) that is independent of the way the base AB is cut into parts h_i and independent of the choice of ordinates y_i on those parts, that limit is defined as the area of the region $ABPQ$.

This summation process provides not only a definition for an area of the above type but also a practical method for calculating its value to any desired approximation.

Example. Find approximately the area bounded by the x-axis, the curve

$$y = \frac{1}{x},$$

and the ordinates at $x = 1$, $x = 2$.

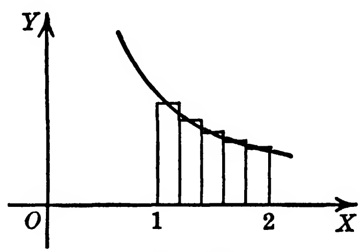

FIGURE 43.

To do this we shall divide the interval from 1 to 2 into 5 parts. There being no particular reason to do otherwise, we make all these parts equal. The parts in question are then determined by consecutive values of x in the following table:

$$x = 1, 1.2, 1.4, 1.6, 1.8, 2.$$

In the strip above each part we are to choose an ordinate to serve as altitude of the approximating rectangle. Since the function

$$y = \frac{1}{x}$$

steadily decreases as x increases, the result will be too large if in each case we take the left-hand value and too small if we take the right. As a

reasonable compromise we take the middle ordinate in each case. The altitudes of the rectangles are then

$$\frac{1}{1.1}, \quad \frac{1}{1.3}, \quad \frac{1}{1.5}, \quad \frac{1}{1.7}, \quad \frac{1}{1.9},$$

and the sum of the rectangles is

$$\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right](0.2) = 0.692.$$

The area correct to three decimals is 0.693.

42. Definite Integral. Many other quantities besides area are represented by limits of essentially the same form as that just described. A limit of this kind has therefore been given a special name and notation defined as follows:

Let $f(x)$ be a function defined for each value of x in the interval (a, b). Divide this interval into n equal or unequal parts by intermediate values $x_2, x_3, \cdots, x_n$ arranged in order from $a = x_1$ to

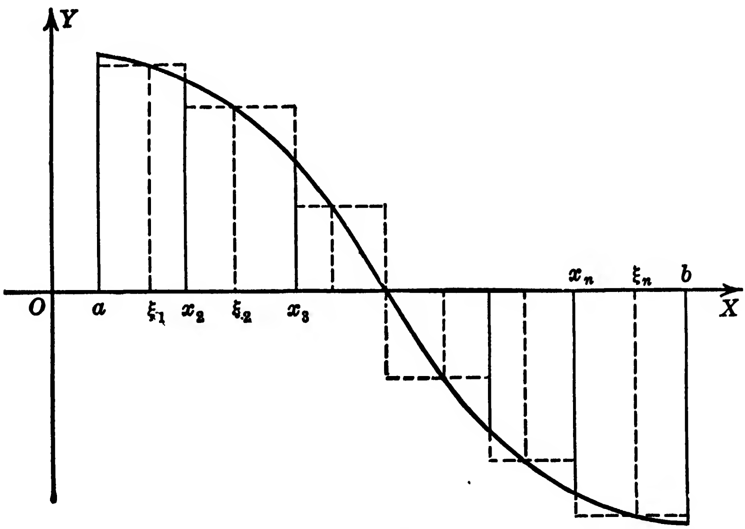

FIGURE 44.

$b = x_{n+1}$. In each part (x_i, x_{i+1}) choose arbitrarily a value ξ_i which may be interior or at either end of this subinterval. Form the sum

$$S_n = f(\xi_1)(x_2 - a) + f(\xi_2)(x_3 - x_2) + \cdots + f(\xi_n)(b - x_n). \quad (1)$$

This sum is often represented by the notation

$$\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i. \tag{2}$$

Consider a sequence of such sums obtained by dividing (a, b) into a larger and larger number of parts but in such a way that the increments

$$\Delta x_i = x_{i+1} - x_i \tag{3}$$

all tend to zero as n tends to infinity. If these sums tend to a limit independent of the way the interval (a, b) is cut into parts and independent of the choice of values $\xi_1, \xi_2, \cdots, \xi_n$ in those parts, the function $f(x)$ is said to be *integrable* in the interval (a, b) and the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i \tag{4}$$

is called the *definite integral* of $f(x)$ between the limits $x = a$ and $x = b$.

It is represented by the notation

$$\int_a^b f(x)\,dx. \tag{5}$$

The number a is called the *lower limit*, b the *upper limit*, of the integral.

If $b > a$ the sum S_n is represented graphically by the area of a set of rectangles, as in Figure 44, those above the x-axis being considered positive, those below negative. The definite integral is thus the difference of the areas above and below the axis.

43. Evaluation of a Definite Integral. To distinguish the definite integral

$$\int_a^b f(x)\,dx$$

from the integral

$$\int f(x)\,dx$$

defined in §36, the latter is often called an *indefinite integral.* The reason for these terms is that the definite integral has a definite value whereas the indefinite integral is any one of a set of functions differing by constants.

The definite and the indefinite integral of a given function have a relation indicated by the following theorem:

If $f(x)$ is continuous and

$$d\,F(x) = f(x)\,dx \tag{1}$$

at each point of the interval (a, b),

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{2}$$

The difference $F(b) - F(a)$ is represented by the notation $[F(x)]_a^b$. The above equation is thus written

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a). \tag{3}$$

To obtain the definite integral of a continuous function $f(x)$ we thus determine any indefinite integral

$$F(x) = \int f(x)\,dx$$

and substitute in the right side of (3).

To prove the theorem, divide the interval (a, b) into parts by values $x_2, x_3, \cdots, x_n$ arranged in order from $a = x_1$ to $b = x_{n+1}$. Since $f(x)$ is the derivative of $F(x)$, by the mean value theorem there is some number ξ_i between x_i and x_{i+1} such that

$$F(x_{i+1}) - F(x_i) = f(\xi_i)(x_{i+1} - x_i) = f(\xi_i)\,\Delta x_i.$$

There are n such equations given by $i = 1, 2, \cdots, n$. Adding these we get

$$F(x_2) - F(a) + F(x_3) - F(x_2) + \cdots + F(b) - F(x_n) = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i,$$

which is equivalent to

$$F(b) - F(a) = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i.$$

Since the left side of this equation is constant the right side tends to that constant as limit when the increments Δx_i all tend to zero. Thus

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i = F(b) - F(a).$$

We have shown this, however, only when the numbers ξ_i have the values determined by the mean value theorem. If in each subinterval (x_i, x_{i+1}) any other value ξ_i' is used, the sum has the form

$$\sum_{i=1}^{n} f(\xi_i')\,\Delta x_i.$$

Since $f(x)$ is continuous and ξ_i, ξ_i' both belong to an interval of length Δx_i we shall have

$$|f(\xi_i') - f(\xi_i)| < \epsilon$$

when all the increments Δx_i are in numerical value less than some number δ [§14, (5)]. Consequently,

$$\begin{aligned} |\Sigma f(\xi_i')\,\Delta x_i - \Sigma f(\xi_i)\,\Delta x_i| &= |\Sigma[f(\xi_i') - f(\xi_i)]\,\Delta x_i| \\ &< \epsilon \Sigma |\Delta x_i| = \epsilon |b - a|. \end{aligned}$$

As the increments Δx_i all tend to zero, δ, and consequently ϵ, may be allowed to approach zero as limit. Thus, whatever positions the numbers ξ_i' have in their respective subintervals (x_i, x_{i+1}), we have

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i')\,\Delta x_i = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i = F(b) - F(a),$$

which was to be proved.

Example. Find the value of

$$\int_0^{\frac{\pi}{2}} \cos x\,dx.$$

By formula V, §38, an indefinite integral of $\cos x$ is

$$\int \cos x\,dx = \sin x.$$

Therefore

$$\int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.$$

44. Area under a Curve. We have seen that, if $b > a$ and $f(x)$ is a positive continuous function, the region bounded by the x-axis, the curve $y = f(x)$, and the ordinates at $x = a$, $x = b$ has the area

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx. \tag{1}$$

If $y = f(x)$ has negative values between a and b, this formula may still be used but the integral will then be equal to the area above the x-axis, minus that below.

An area which does not have boundaries of exactly the form stated can usually be expressed as a sum or difference of such areas.

Example 1. Find the area bounded by the x-axis, the curve $y = 1 + x^2$, and the ordinates at $x = -1$, $x = 1$.

The required area is shown in Figure 45. By equation (1) it is

$$A = \int_{-1}^{1} (1 + x^2)\, dx = [x + \tfrac{1}{3}x^3]_{-1}^{1} = \tfrac{4}{3} - (-\tfrac{4}{3}) = \tfrac{8}{3}.$$

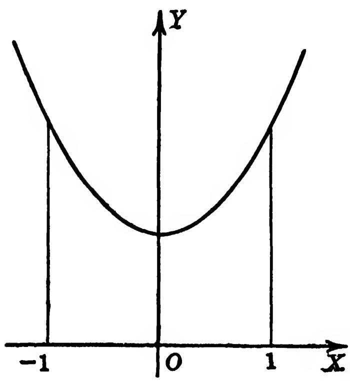

FIGURE 45.

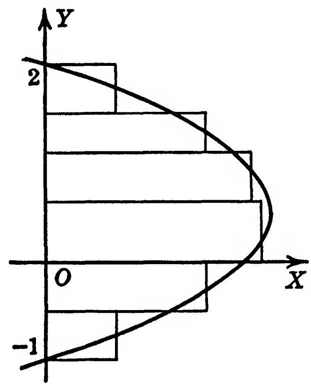

FIGURE 46.

Example 2. Find the area bounded by the curve $x = 2 + y - y^2$ and the y-axis.

The curve (Figure 46) crosses the y-axis at $x = 2 + y - y^2 = 0$, that is, at $y = -1$, $y = 2$. Since the area is bounded by the curve and the y-axis it is the limit of a set of rectangles of sides x and Δy extending from $y = -1$ to $y = 2$. Its value is therefore

$$\lim_{\Delta y \to 0} \sum_{-1}^{2} x\Delta y = \int_{-1}^{2} (2 + y - y^2)\, dy$$

$$= [2y + \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3]_{-1}^{2} = 4\tfrac{1}{2}.$$

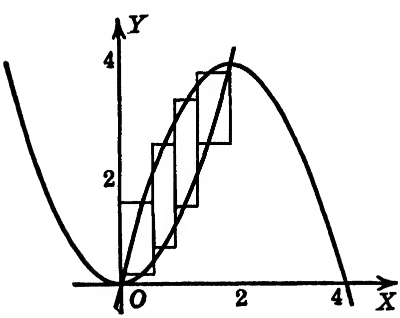

FIGURE 47.

Example 3. Find the area bounded by the curves

$$y = x^2,\ y = 4x - x^2.$$

Solving the equations simultaneously, we find that the two curves

intersect at $x = 0$ and $x = 2$. Taking $y_1 = x^2$, $y_2 = 4x - x^2$, the area is the limit of a set of rectangles of base Δx and altitude $y_2 - y_1$, extending from $x = 0$ to $x = 2$. Thus

$$A = \int_0^2 (y_2 - y_1)\,dx = \int_0^2 (4x - 2x^2)\,dx = \left[2x^2 - \tfrac{2}{3}x^3\right]_0^2 = \tfrac{8}{3}.$$

45. Volume of a Solid of Revolution. The region bounded by the x-axis, a curve $y = f(x)$, and two ordinates $x = a$, $x = b$ is rotated about the x-axis. We wish to determine the volume of the solid generated.

To do this divide the interval (a, b) into equal or unequal parts and on each part $(x, x + \Delta x)$ construct a rectangle of base Δx and altitude equal to the ordinate $f(\xi)$ at some one of its points $x = \xi$.

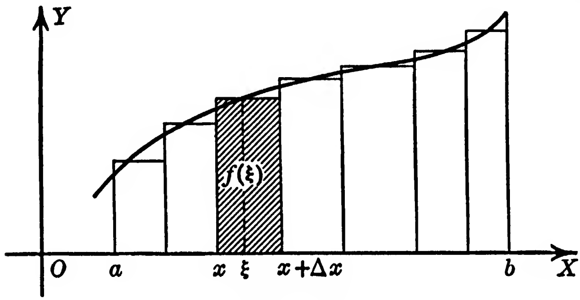

FIGURE 48.

When rotated about the x-axis this rectangle generates a cylinder of radius $f(\xi)$, altitude Δx, and volume

$$\pi[f(\xi)]^2\,\Delta x.$$

The volume generated by the set of rectangles is

$$\sum_a^b \pi[f(\xi)]^2\,\Delta x.$$

If $f(x)$ is continuous between a and b, when the number of divisions increases and all the increments Δx tend to zero, this sum tends to the limit

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_a^b \pi y^2\,dx, \tag{1}$$

which is defined as the volume of the region generated.

This formula shows that the volume of the solid is numerically equal to the area between the x-axis and the curve

$$y = \pi\,[f(x)]^2.$$

Similarly, each application of the definite integral is made essentially by showing that the quantity in question is a limit of the same form as that which defines a certain area. For this reason the determination of a quantity by definite integration is often called *quadrature.*

If a solid is generated by rotating an area, not about the x-axis but about some other line, and all cross sections perpendicular to the axis are circles, its volume is

$$V = \int_a^b \pi R^2\, dh,$$

where h is distance measured along the axis, R the radius of the cross section at position h, and a, b limiting values of h.

Example 1. The area bounded by the x-axis and a curve

$$y = 2x - x^2$$

is rotated about the x-axis. Find the volume generated.

The curve crosses the x-axis at $x = 0$ and $x = 2$. The volume required is therefore

$$\int_0^2 \pi y^2\, dx = \int_0^2 \pi(4x^2 - 4x^3 + x^4)\, dx = \pi[\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5]_0^2 = \tfrac{16}{15}\pi.$$

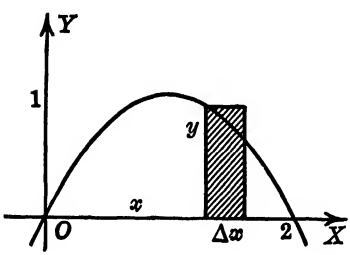

FIGURE 49.

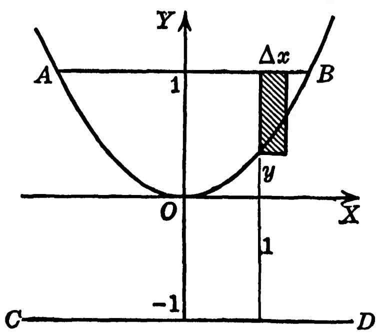

FIGURE 50.

Example 2. The area bounded by the curve $y = x^2$ and the line $y = 1$ is revolved about the line $y = -1$. Find the volume generated.

The area between the curve AOB (Figure 50) and the line AB is revolved about the line CD. This area is cut into strips by lines parallel to the y-axis. The strip between x and $x + \Delta x$ is approximately a rectangle of base Δx and altitude $1 - y$. Rotated about the line CD this generates a hollow cylinder, or washer of inner radius $y + 1$, outer radius 2, thickness Δx, and volume

$$\pi[2^2 - (y + 1)^2]\, \Delta x = \pi(3 - 2y - y^2)\, \Delta x.$$

The entire volume is the limit of the sum of such cylinders extending from $x = -1$ to $x = 1$. It is therefore

$$\int_{-1}^{1} \pi(3 - 2y - y^2)\,dx = \pi\int_{-1}^{1} (3 - 2x^2 - x^4)\,dx$$
$$= \pi\left[3x - \tfrac{2}{3}x^3 - \tfrac{1}{5}x^5\right]_{-1}^{1}$$
$$= \tfrac{64}{15}\pi.$$

46. Volume of a Solid with Given Area of Section. Consider a solid cut by planes perpendicular to an axis along which a coordinate h is measured. Let $A(h)$ be the area of section in the

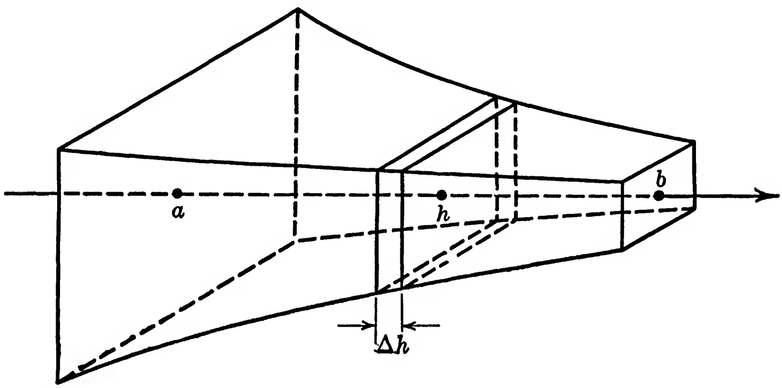

FIGURE 51.

plane which crosses the axis at position h. Divide the portion of the axis between $h = a$ and $h = b$ into equal or unequal parts, and pass planes perpendicular to the axis through the points of division. The slice between the planes at h, $h + \Delta h$ is approximately a plate (not necessarily circular) of face area $A(h)$, thickness Δh, and volume $A(h)\,\Delta h$. The entire solid is approximately a sum of such plates of total volume

$$\sum_a^b A(h)\,\Delta h.$$

When all the increments Δh tend to zero this sum approaches the volume of the solid

$$V = \int_a^b A(h)\,dh$$

as limit.

To determine the volume of such a solid we thus merely express the area of section $A(h)$ in terms of h and integrate between the appropriate limits.

Example. The axes of two equal right circular cylinders intersect at right angles. Find the common volume.

Let a be the radius, OY and OZ the axes of the cylinders, and OX a line perpendicular to both axes. In Figure 52, $OABC$ is one-eighth of the

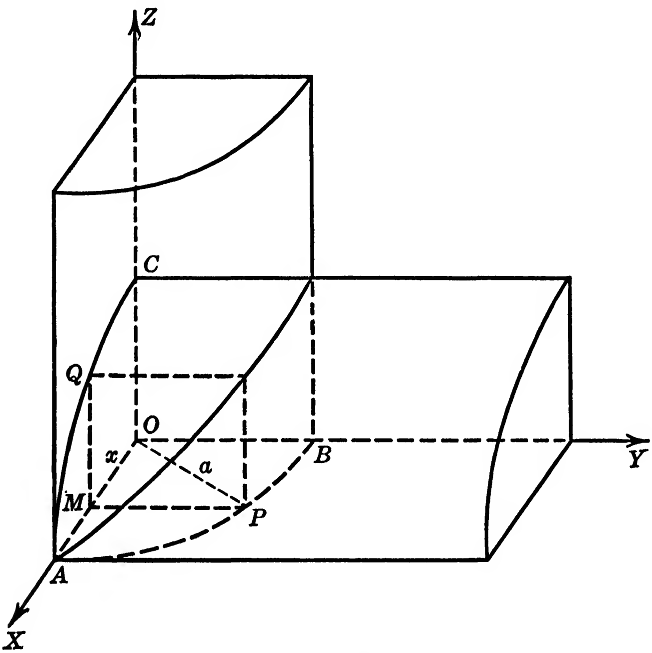

FIGURE 52.

common volume and PMQ is a section perpendicular to OX at distance x from the center. Since OP, OQ are radii and OMP, OMQ right triangles

$$MP = MQ = \sqrt{a^2 - x^2}.$$

Thus the section PMQ is a square of area

$$A(x) = MP \cdot MQ = a^2 - x^2.$$

The volume common to the two cylinders is therefore

$$V = 8\int_0^a A(x)\,dx = 8\int_0^a (a^2 - x^2)\,dx = \tfrac{16}{3}a^3.$$

47. Pressure. A fluid at rest in contact with one side of a flat plate exerts a force perpendicular to the surface of the plate. If ΔF is the magnitude of this force on an area ΔA, the limit

$$p = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}, \tag{1}$$

as ΔA shrinks to zero about a point P, is called the force per unit area, or *pressure*, at P. The pressure at a point is independent of the direction of the surface across which it is measured and in a liquid of constant weight w per unit volume has the magnitude

$$p = wh \tag{2}$$

at depth h below the surface.

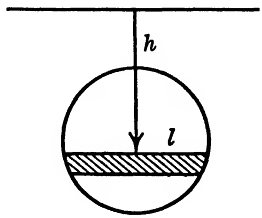

FIGURE 53.

We wish to determine the force exerted by liquid pressure against one side of a vertical plate such as a gate in the face of a dam. To do this divide the plate into strips by horizontal lines and let l be the width and p the pressure at depth h below the surface. The area between the lines at depths h, $h + \Delta h$ is approximately $l\,\Delta h$ and the force on this approximately $pl\,\Delta h$, these approximations becoming more accurate as Δh diminishes. The total force is thus

$$F = \lim_{\Delta h \to 0} \Sigma pl\,\Delta h = \int_a^b pl\,dh = w\int_a^b hl\,dh, \tag{3}$$

a, b being the values of h at the top and bottom of the submerged area.

To determine the force in a particular case we express l in terms of h, or l and h in terms of a third variable, and integrate (3) between the limiting values of the variable used in the integration.

In the case of water

$$\begin{aligned} w &= 62.5 \text{ lb./cu. ft.} \qquad (4)\\ &= \tfrac{1}{32} \text{ ton/cu. ft.,} \end{aligned}$$

approximately.

Example 1. A vertical gate in a dam has the form of a square of side 4 feet, the upper edge being 2 feet below the surface of the water. Find the force it must withstand.

In this case $l = 4$ and h varies from 2 feet to 6 feet. Hence

$$F = \int_2^6 wh \cdot 4\, dh = [2wh^2]_2^6 = 2 \text{ tons.}$$

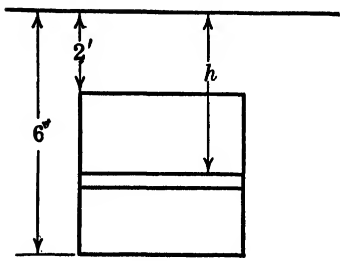

FIGURE 54.

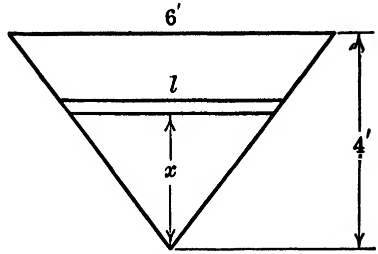

FIGURE 55.

Example 2. Find the total pressure on one side of a triangle of base 6 feet and altitude 4 feet if its altitude is vertical and base in the surface of the water.

At distance x above the vertex the width of the triangle (Figure 55) is

$$l = \tfrac{6}{4}x = \tfrac{3}{2}x.$$

Also x varies from 0 to 4, $h = 4 - x$. Thus

$$F = \int_0^4 w(4 - x) \cdot \tfrac{3}{2}x\, dx = w[3x^2 - \tfrac{1}{2}x^3]_0^4 = 1000 \text{ lb.}$$

48. Work. Let a force F be applied at a fixed point of a body. When the body moves, work is done by the force. At present we consider only motion along a straight line under the action of a force parallel to the line; motion in a curve will be considered later (§200).

If the force is constant, the work w done by the force is the product

$$W = Fs$$

where F is the force and s the distance moved along the direction of the force. The work done by a force is positive when the displacement is in the direction of the force, negative when in the opposite direction.

If the force is variable, represent the position of the body by its displacement x from a fixed point of its line of motion. Consider the force positive when in the direction in which x increases, and let $F(x)$ be its value when the displacement is x. In a small motion Δx the force is nearly constant and the work approximately

$F(x)\,\Delta x$. As Δx diminishes, this approximation becomes more accurate. The exact work done by the force $F(x)$ is thus

$$W = \lim_{\Delta x \to 0} \Sigma F(x)\,\Delta x = \int_a^b F(x)\,dx \tag{1}$$

when the body moves from $x = a$ to $x = b$.

When the force is expressed in pounds and the displacement in feet the work is given in foot-pounds.

Example 1. The amount a helical spring is stretched is proportional to the force applied. If a force of 100 pounds stretches the spring 1 inch, find the work done in stretching it 4 inches.

Let x be the number of inches the spring is stretched. The force is then

$$F = kx,$$

k being constant. When $x = 1$, $F = 100$. Hence

$$100 = k$$

and

$$F = 100x.$$

The work done in stretching the spring 4 inches is thus

$$\int_0^4 F\,dx = \int_0^4 100x\,dx = 800 \text{ in.-lb.} = 66\tfrac{2}{3} \text{ ft.-lb.}$$

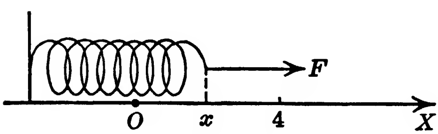

FIGURE 56.

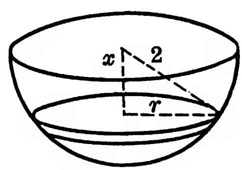

FIGURE 57.

Example 2. A hemispherical tank of radius 2 feet is full of water weighing 62.5 pounds per cubic foot. How much work is required to pump the water from the tank?

Divide the water into slices by horizontal planes (Figure 57). The volume of water between the planes at depths x, $x + \Delta x$ below the center is approximately a cylinder of radius

$$r = \sqrt{4 - x^2},$$

altitude Δx, volume

$$\pi r^2\,\Delta x = \pi(4 - x^2)\,\Delta x$$

and weight

$$62.5\pi(4 - x^2)\,\Delta x \text{ lb.}$$

The work done in lifting this slice the distance x to the top is thus approximately

$$x[62.5\pi(4 - x^2)\,\Delta x] \text{ ft.-lb.}$$

The total work required to empty the tank is therefore

$$\int_0^2 62.5\pi x(4 - x^2)\,dx = 250\pi \text{ ft.-lb.}$$

49. Properties of Definite Integrals. If $f(x)$ is continuous for the values of x involved, the following equations are almost immediate consequences of the definition of definite integral:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx, \tag{1}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{2}$$

The first of these equations states that interchange of limits changes the algebraic sign of a definite integral. To prove this we notice that if the same intermediate values are used in both cases the increments Δx when x varies from b to a merely differ in algebraic sign from those obtained when x varies from a to b. Thus

$$\sum_b^a f(\xi)\,\Delta x = -\sum_a^b f(\xi)\,\Delta x,$$

from which (1) follows by taking limits.

If c is between a and b, equation (2) expresses graphically that the area between the ordinates at a and b (Figure 58) is equal to the

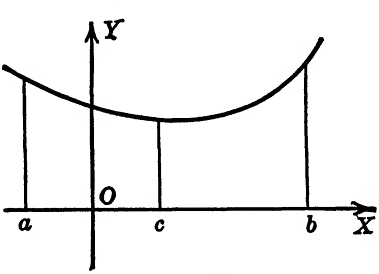

FIGURE 58.

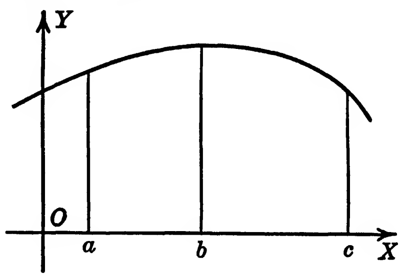

FIGURE 59.

area between a and c plus that between c and b. It should be noted, however, that this equation is also valid when c is not between a and b. In Figure 59, for example,

$$\int_c^b f(x)\,dx$$

is negative and so

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

is equal to the area from a to c minus that from b to c, the result being

$$\int_a^b f(x)\,dx.$$

50. Mean Value Theorem. *If $f(x)$ is continuous in the interval (a, b), there is some value ξ between a and b such that*

$$\int_a^b f(x)\,dx = (b - a)f(\xi). \tag{1}$$

To prove this we consider $b > a$. The case $b < a$ can be handled in a similar way. Let m be the number such that

$$\int_a^b f(x)\,dx = (b - a)m. \tag{2}$$

Then

$$\int_a^b [f(x) - m]\,dx = 0. \tag{3}$$

This shows that $f(x) - m$ cannot be positive for all values of x between a and b, for the integral would then be positive. Similarly $f(x) - m$ cannot be negative for all these values. This function must then either be zero for all such values or be negative for some and positive for others. In the latter case there will be at least one value between a place where $f(x) - m$ is positive and a place where $f(x) - m$ is negative at which this function is zero (intermediate value theorem, §14). In any case there is then at least one value $x = \xi$ between a and b such that

$$f(\xi) - m = 0,$$

and (1) follows from (2).

51. Integral with Variable Limit. If $f(x)$ is continuous for the values considered and a is constant, the integral

$$F(x) = \int_a^x f(t)\,dt \tag{1}$$

is a function of the upper limit x. Graphically it represents the area (Figure 60) bounded by the x-axis, the curve $y = f(x)$, a fixed

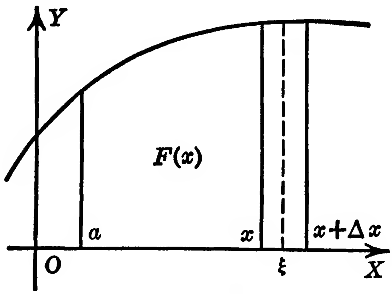

FIGURE 60.

ordinate at $x = a$, and the variable ordinate of abscissa x. We shall show that this function has the derivative

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

To prove this we first determine the increment

$$\Delta F(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+\Delta x} f(t)\,dt.$$

If $f(t)$ is continuous at $t = x$, by the mean value theorem (§50)

$$\int_x^{x+\Delta x} f(t)\,dt = \Delta x f(\xi),$$

where ξ is between x and $x + \Delta x$. Thus

$$\frac{\Delta F(x)}{\Delta x} = f(\xi).$$

When Δx tends to zero, ξ tends to x, and this equation gives as limit

$$\frac{dF(x)}{dx} = f(x),$$

which was to be proved.

PROBLEMS

Find the values of the following integrals:

1. $\int (3x^2 - 4x + 2)\,dx$. **2.** $\int (8x^3 + 6x^2 - 2x + 5)\,dx$.

3. $\int \left(y^2 + \frac{1}{y^2}\right) dy$. **4.** $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$.

5. $\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right) dx.$

6. $\int (2t + 1)^2\, dt.$

7. $\int \frac{x^3 - 2x^2 - 3}{x^2}\, dx.$

8. $\int (x - 1)(x - 2)\, dx.$

9. $\int x(x + 1)^2\, dx.$

10. $\int \frac{du}{(u + 1)^2}.$

11. $\int \frac{dx}{(3x + 2)^2}.$

12. $\int \sqrt{2t + 3}\, dt.$

13. $\int \frac{dx}{\sqrt{3x + 2}}.$

14. $\int \frac{y\, dy}{\sqrt{y^2 - 1}}.$

15. $\int \frac{x\, dx}{(3x^2 + 2)^2}.$

16. $\int x\sqrt{3x^2 + 2}\, dx.$

17. $\int \frac{x^2\, dx}{(a^3 + x^3)^2}.$

18. $\int x^3\sqrt{4x^4 + 5}\, dx.$

19. $\int \frac{(2x + 1)\, dx}{\sqrt{x^2 + x + 1}}.$

20. $\int \left(1 + \frac{1}{t}\right)^2 \frac{dt}{t^2}.$

21. $\int \cos 2\theta\, d\theta.$

22. $\int \sin (2x + 3)\, dx.$

23. $\int \sin t \cos t\, dt.$

24. $\int \sqrt{1 - \cos y}\, \sin y\, dy.$

25. A body, started vertically downward with a velocity of 30 feet per second, falls under the acceleration of gravity. How far will it fall in t seconds?

26. From a height h_0 above the ground a body is started upward with a speed v_0. If it is subject only to the acceleration g of gravity, find its height at the end of t seconds.

27. If the velocity of a body at time t is

$$v = (2t + 3t^2) \text{ ft./sec.},$$

find the distance traversed between $t = 2$ and $t = 5$.

28. A body, starting at the origin and moving along the x-axis, has the velocity

$$\frac{dx}{dt} = \pi \sin \frac{\pi t}{2} \text{ ft./sec.}$$

at the end of t seconds. Find its position when $t = 2$.

29. A particle, starting from rest, has the acceleration

$$(4t^2 - 2t + 8) \text{ ft./sec.}^2$$

at the end of t seconds. Find its average velocity during the first 3 seconds.

30. The velocity of a body is

$$v = \pi \cos \frac{\pi}{4} t \text{ ft./sec.}$$

at the end of t seconds. Find its average velocity during the first 2 seconds.

31. Assuming that the brakes of an automobile produce a constant negative acceleration $-k$ feet per second per second, find what k must be if when trav-

eling 60 miles per hour the automobile is brought to rest in a distance of 100 feet.

32. At each point (x, y) of a curve the slope is

$$\frac{dy}{dx} = 2x - 3.$$

If the curve passes through (2, 3), find its equation.

33. At each point (x, y) of a curve the slope is $\sin x$. If the curve passes through (0, 1), find its equation.

34. At each point (x, y) of a curve the slope is kx. Find the value of k if the curve passes through (0, 0) and (−1, 2).

35. On a certain curve

$$\frac{d^2y}{dx^2} = x - 1.$$

If the curve passes through (−1, 1) and has at that point the slope 2, find its equation.

36. At each point (x, y) of a curve

$$\frac{d^2y}{dx^2} = 6x.$$

If the curve passes through (1, 2) and (0, −1), find its equation.

37. By dividing the interval into 5 parts determine approximately the area bounded by the x-axis, the curve $y = x^2$, and ordinates at $x = 0$, $x = 1$. The exact area is $\frac{1}{3}$.

38. By dividing the interval into 5 parts determine approximately the area bounded by the x-axis, the curve

$$y = \frac{1}{1 + x^2},$$

and ordinates at $x = 0$, $x = 1$. The exact area is $\frac{1}{4}\pi$.

39. By dividing the area into 6 parts determine approximately the area bounded by the x-axis, the curve

$$y = \sqrt{x(12 - x)},$$

and ordinates at $x = 0$, $x = 12$. The exact area is 18π.

40. By dividing the area into 6 parts determine approximately the area bounded by the x-axis, the curve $y = 2 \sin x$, and ordinates at $x = 0$, $x = \frac{\pi}{2}$. The exact area is 2.

Find the values of the following definite integrals:

41. $\int_0^3 (x^2 - 2x + 2)\, dx.$

42. $\int_{-1}^1 (1 - x^2)\, dx.$

43. $\int_3^8 \sqrt{x + 1}\, dx.$

44. $\int_0^\pi \sin x\, dx.$

45. $\int_{-\pi}^\pi \cos x\, dx.$

46. $\int_0^a x\sqrt{a^2 - x^2}\, dx.$

47. Find the area bounded by the x-axis, the line $y = 2x - 1$, and the ordinates at $x = 1$, $x = 2$.

48. Find the area bounded by the coördinate axes, the curve

$$y = 3x^2 + 4x + 1,$$

and the line $x = 2$.

49. Find the area bounded by the x-axis, the curve $y = \sin x$, and the ordinates at $x = 0$, $x = \frac{\pi}{3}$.

50. Find the area bounded by the x-axis and the curve

$$y = 4 - x^2.$$

51. Find the area bounded by the x-axis and the curve

$$y = 4x - x^2 - 3.$$

52. Find the area bounded by the y-axis, the curve

$$x = 3y^2 + 2,$$

and the lines $y = -1$, $y = 2$.

53. Find the area bounded by the y-axis and the curve

$$x = 2y - y^2.$$

54. Find the area bounded by

$$y^3 = x$$

and the lines $x = 1$, $y = -1$.

55. Find the area bounded by the curve

$$y = 2x - x^2$$

and the line $y = -3$.

56. Find the area bounded by $y = 4 - x^2$ and $y = 8 - 2x^2$.

57. Find the area bounded by $y = x^3 + x^2$, $y = x^3 + 1$.

58. Find the area bounded by $y = x - x^2$ and $y = -x$.

59. Find the area bounded by the straight line $x + 2y = 3$ and the curve $x = y^2 - 3y + 1$.

60. The triangle bounded by the coördinate axes and the line

$$y = 2 - x$$

is rotated about the x-axis. Find the volume generated.

61. The area bounded by the x-axis and the curve

$$y = 4 - x^2$$

is rotated about the x-axis. Find the volume generated.

62. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is rotated about the x-axis. Find the volume generated.

63. The area above the x-axis, bounded by the curve $y^3 = x$ and the line $x = 1$, is rotated about the x-axis. Find the volume generated.

64. A sphere of radius a is generated by rotating the circle

$$x^2 + y^2 = a^2$$

about the x-axis. Find its volume by integration.

65. The area within the circle $x^2 + y^2 = a^2$ is cut into two parts by the line $x = h$. Find the volumes of the spherical segments generated by rotating these parts about the x-axis.

66. Find the volume of the ellipsoid generated by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

67. The area bounded by the y-axis and the curve

$$x = 2y - y^2$$

is rotated about the y-axis. Find the volume generated.

68. The area bounded by the curve $y = 2 - x^2$ and the line $y = 1$ is rotated about the x-axis. Find the volume generated.

69. The area bounded by the curve $y^2 = 4ax$ and the line $x = a$ is rotated about the line $x = a$. Find the volume generated.

70. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is rotated about the y-axis. Find the volume generated.

71. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is rotated about the line $x = 2a$. Find the volume generated.

72. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is rotated about the line $y = -2a$. Find the volume generated.

73. The area bounded by the hyperbola $xy = 4$ and the line $x + y = 5$ is rotated about the y-axis. Find the volume generated.

74. Sections of a solid parallel to a fixed plane are circles, the section at distance x from the plane having the radius $r = x^2$. Find the volume of the solid between the sections at $x = a$ and $x = b$.

75. The base of a solid is a circle of radius r and diameter AB, and each cross section perpendicular to AB is a square. Find the volume of the solid.

76. The base of a solid is the area bounded by the parabola $y^2 = 4ax$ and the line $x = a$, and each cross section perpendicular to the x-axis is an equilateral triangle. Find the volume of the solid.

77. The base of a solid is the area bounded by the x-axis and the lines $y = 2x$, $x = 2$, and each cross section perpendicular to the y-axis is a triangle of altitude 3. Find the volume of the solid.

78. A square of side a revolves about a line perpendicular to its plane while the point of intersection of line and plane moves the distance h along the line. Find the volume of the twisted solid which is generated.

79. Two circles have a common diameter and lie in perpendicular planes. A square moves in such a way that its plane is perpendicular to this diameter and its diagonals are chords of the circles. Find the volume generated.

80. The base of a solid is a circle of radius a and diameter AB. Each section perpendicular to AB is a triangle of altitude h. Find the volume of the solid. To evaluate the integral that is obtained observe that it is a constant times the area of the circle.

81. A wedge is cut from the base of a right circular cylinder by a plane passing through a diameter of the base and inclined at the angle α to the base. Find the volume of the wedge.

82. The generators of an oblique cylinder make an angle of 45° with the base. A wedge is cut from the base of the cylinder by a plane through the

center of the base and perpendicular to the generators. Find the volume of the wedge.

83. Find the force due to water pressure on a rectangular flood gate 10 feet broad and 8 feet deep, the upper edge being in the surface of the water.

84. Find the force on the lower half of the gate in the preceding problem.

85. Find the force due to water pressure on one side of a triangle of base b and altitude h submerged so that its vertex is in the surface of the water and its altitude vertical.

86. A vertical dam is in the form of a trapezoid 300 feet long at the surface, 200 feet at the bottom, and 30 feet high. Find the horizontal force on the dam when the water is 20 feet deep.

87. The end of a pool is a rectangle inclined 45° to the vertical. The edge in the surface of the water is 12 feet long, and the submerged edge, 10 feet. Find the force due to fluid pressure on the rectangle.

88. According to Hooke's law the force required to stretch an elastic rod of natural length a to the length $a + x$ is

$$\frac{kx}{a},$$

where k is constant. Find the work done in stretching the rod from the length a to the length b.

89. A cylindrical cistern, of diameter 4 feet and depth 8 feet, is full of water. Find the work required to pump the water out.

90. A tank, 20 feet in diameter and 16 feet deep, is half full of water. Find the work required to pump the water out.

91. A pyramid, of altitude h feet and base A square feet, is built of rock of weight w pounds per cubic foot. Find the work required to lift the stones from the ground into place.

92. Assuming that the force of gravity on a mass m at distance r from the center of the earth is

$$\frac{km}{r^2}$$

where k is constant, find the work done by gravity on a 10-pound body when it moves from an indefinitely great distance to the surface of the earth. Take the radius of the earth as 4000 miles and use the known force of gravity at the surface of the earth to calculate k.

93. Inside the earth a particle is attracted toward the center with a force proportional to the distance from the center. Find the work required to lift a 10-pound body from the center to the surface of the earth.

94. A weight of 500 pounds hangs at the end of a 200-foot chain which weighs 2 pounds per foot. Find the work done in winding in the chain.

95. A gas is compressed in a cylinder by a movable piston. Assuming that it is compressed adiabatically (that is, without receiving or giving out heat), its pressure and volume satisfy the equation

$$pV^\gamma = \text{constant},$$

where γ for air is about 1.4. Find the work done in the adiabatic compression of 100 cubic inches of air from atmospheric pressure (14.7 pounds per square inch) to a volume of 10 cubic inches.

CHAPTER IV

ALGEBRAIC EQUATIONS AND GRAPHS

52. Equation and Locus. To each equation

$$f(x, y) = 0$$

corresponds a locus consisting of all points with rectangular coördinates x, y which satisfy the equation. The locus is called the graph of the equation, and the equation is said to represent the locus. Since rectangular coördinates were first used by Descartes, the rectangular equation of a locus is often called its *Cartesian* equation.

To each analytical property of such an equation or set of equations corresponds a geometrical property of the locus or loci represented. The study of this correspondence between analytical and geometrical properties and, in particular, the deduction of geometrical relations by means of the corresponding analytical characteristics constitutes a subject called analytical geometry.

In this chapter we shall consider certain algebraic equations in x and y and the plane loci they represent. To describe briefly what we mean by an algebraic equation, consider first a single term

$$ax^m y^n,$$

where a is constant and m, n positive whole numbers. This term is said to be of degree $m + n$ in x and y. The sum of a finite number of such terms, whether of the same or different degrees, is called a polynomial. The degree of the polynomial is defined as that of its terms of highest degree. The equation obtained by equating to zero an nth degree polynomial is called an algebraic equation of the nth degree.

Thus

$$x^2 - xy + 3y + 7 = 0$$

is an equation of the second degree and

$$y^3 - 2x = 0$$

is one of the third degree.

53. Point of Division. Along a line not parallel to a coördinate axis no positive direction is usually assigned. In handling distances along such a line it is convenient, however, to consider a segment P_1P_2 as described from P_1 to P_2 and to consider two segments as having the same algebraic sign when described in the same direction, different signs when described in opposite directions.

To find the point dividing a line segment in a given ratio we project on the coördinate axes and use the fact (which follows from similar triangles) that the projections have the same ratio.

Example 1. Find the point $P(x, y)$ on the line through $P_1(1, -2)$, $P_2(7, 4)$ two-thirds of the way from P_1 to P_2.

Let M_1, M, M_2 be the projections of P_1, P, P_2 on the x-axis (Figure 61). Then M is two-thirds of the way from M_1 to M_2. The coördinate x of M is the displacement in moving from the origin to M. This may be obtained as the sum of the displacements in going from the origin to M_1 and then two-thirds of the way from M_1 to M_2. That is,

$$x = 1 + \tfrac{2}{3}(7 - 1) = 5.$$

Similarly,

$$y = -2 + \tfrac{2}{3}(4 + 2) = 2.$$

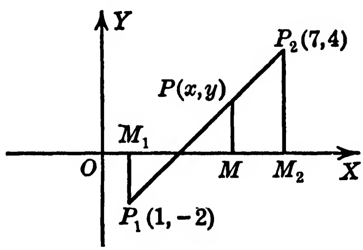

FIGURE 61.

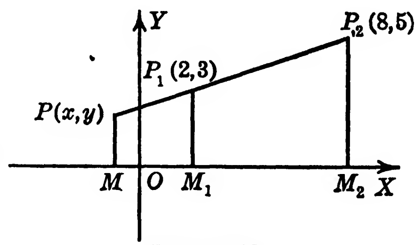

FIGURE 62.

Example 2. Find the point on the line through $P_1(2, 3)$, $P_2(8, 5)$ such that

$$P_1P = -\tfrac{1}{2}P_1P_2.$$

By this we mean that the points are related as in Figure 62. The x-coördinate of P may be obtained as the displacement from the origin to M_1 followed by a displacement $-\frac{1}{2}M_1M_2$. Thus

$$x = 2 - \tfrac{1}{2}(8 - 2) = -1.$$

Similarly,

$$y = 3 - \tfrac{1}{2}(5 - 3) = 2.$$

54. Equation of a Straight Line. Let $P_1(x_1, y_1)$ be a fixed point and $P(x, y)$ a variable point on a given line LM.

If the line is not perpendicular to the x-axis its slope is

$$m = \frac{y - y_1}{x - x_1},$$

whence

$$y - y_1 = m(x - x_1). \tag{1}$$

This equation is satisfied by the coördinates of any point (x, y) on the line and by those of no other point. It is therefore *the equation of the line through* (x_1, y_1) *with slope equal to m.*

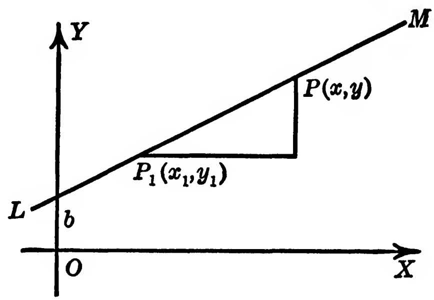

FIGURE 63.

In particular, if we take (x_1, y_1) as the point $(0, b)$ where the line crosses the y-axis, (1) becomes

$$y = mx + b. \tag{2}$$

The number b is called the intercept of the line on the y-axis. Thus (2) *is the equation of the line with slope m and intercept b.*

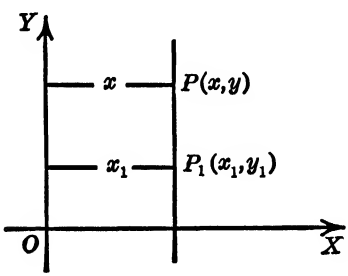

FIGURE 64.

If the line is perpendicular to the x-axis, its slope is infinite and equations (1) and (2) cannot be used. In that case a diagram (Figure 64) shows, however, that

$$x = x_1 \tag{3}$$

is the equation of the line.

Example. Find the equation of the line through the points (1, 2) and (3, −4).

The slope of the line is

$$m = \frac{-4 - 2}{3 - 1} = -3.$$

Since the line passes through (1, 2) and has this slope, its equation is

$$y - 2 = -3(x - 1),$$

which is equivalent to

$$y + 3x = 5.$$

55. First-Degree Equation. Equations (1), (2), (3), of §54 are of the first degree in x and y. Thus *any straight line is represented by an equation of the first degree in rectangular coördinates.*

Conversely, *any equation of the first degree in rectangular coordinates represents a straight line.* For any equation of the first degree in x and y has the form

$$Ax + By + C = 0, \tag{1}$$

A, B, C being constants. If B is not zero, this is equivalent to

$$y = -\frac{A}{B}x - \frac{C}{B},$$

which represents the line [§54, (2)] with slope $-\frac{A}{B}$ and intercept $-\frac{C}{B}$ on the y-axis. If B is zero the equation is equivalent to

$$x = -\frac{C}{A},$$

which represents a line perpendicular to the x-axis [§54, (3)].

Because of this relation to a straight line a first-degree equation in x and y is called *linear*. By analogy the range of this term has been extended to include all first-degree equations.

A linear equation in x and y appears to contain three constants A, B, C. Since, however, division by a constant does not change the equation, it really contains only two significant constants, for instance, $\frac{A}{B}$ and $\frac{C}{B}$. This means algebraically that the constants in the equation are essentially determined by two equations and geometrically that the line may be made to fulfill two conditions. For example, a line can be made to pass through two points, to pass through one point and be perpendicular to a second line, to be tangent to two circles, etc.

If the coefficient B is zero, the line is perpendicular to the x-axis and its slope is infinite. If B is not zero, the equation can be solved for y and so reduced to the form,

$$y = mx + b.$$

The slope is then the coefficient of x.

56. Angles. An angle is regarded as generated or described by rotation around its vertex from an initial to a terminal side, the direction of description often being indicated by an arrow. The angle is considered positive if its direction is that which carries OX through 90° into OY. With the usual positions of the axes this amounts to saying that an angle is positive when it is described in the counterclockwise direction, negative when in the clockwise direction. In Figure 65, for example, the angle A extends from

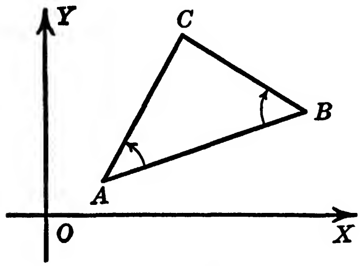

FIGURE 65.

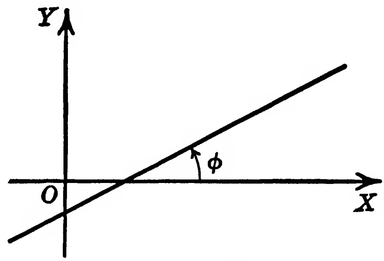

FIGURE 66.

AB to AC and is positive, whereas B extends from BA to BC and is negative.

The direction of a line is usually determined by means of the angle ϕ extending from the positive end of the x-axis to the line. This angle may be drawn in either direction and its terminal side may be either end of the line. Any one of these angles uniquely determines the direction of the line and any one of them satisfies the equation

$$\tan \phi = m, \tag{1}$$

m being the slope of the line.

In the following diagrams ϕ is taken as the smallest positive one of these angles. But in some cases, as for instance when the line is variable, this restriction may not prove convenient.

Let ϕ_1, ϕ_2 be the angles which two lines make with the x-axis, and

$$m_1 = \tan \phi_1, \quad m_2 = \tan \phi_2 \tag{2}$$

their slopes.

If the lines are parallel (Figure 67) the angles ϕ_1, ϕ_2 are equal and the slopes are equal. Conversely, if the slopes are equal the lines are parallel. Thus, *two lines are parallel if and only if their slopes are equal.*

If the two lines are perpendicular, ϕ_2 and ϕ_1 differ by 90° and

$$\tan \phi_2 = -\cot \phi_1 = -\frac{1}{\tan \phi_1}.$$

Thus

$$m_2 = -\frac{1}{m_1}. \tag{3}$$

Conversely, if this equation is satisfied, $\tan \phi_2 = -\cot \phi_1$ and the

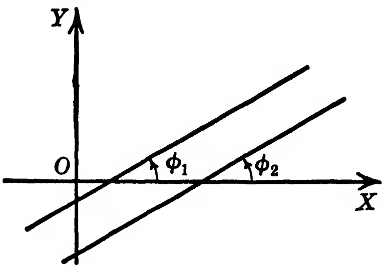

FIGURE 67.

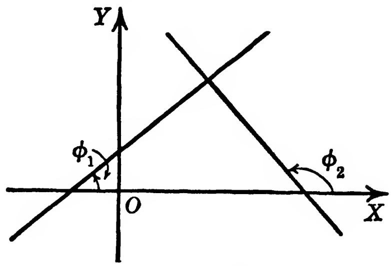

FIGURE 68.

angles differ by 90°. Thus, *two lines are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other.*

Let

$$\beta = \phi_2 - \phi_1 \tag{4}$$

be the angle from the line of slope m_1 to that of slope m_2. Then

$$\tan \beta = \tan (\phi_2 - \phi_1) = \frac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_1 \cdot \tan \phi_2}.$$

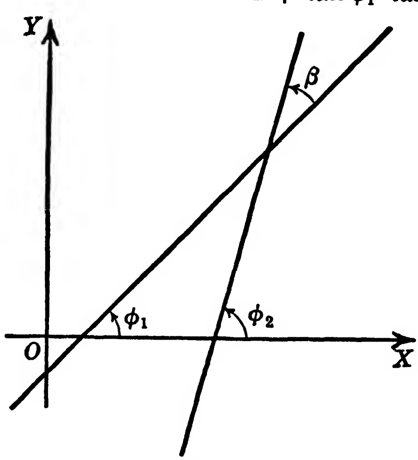

FIGURE 69.

Since $\tan \phi_1 = m_1$, $\tan \phi_2 = m_2$, this is equivalent to

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{5}$$

The angle β in this equation may be any angle extending from the line of slope m_1 to that of slope m_2. To find the tangent of a particular angle take m_1 as the slope of its initial side and m_2 as that of its terminal side.

Example 1. Show that the lines $3x - 4y + 1 = 0$, $4x + 3y - 5 = 0$, are perpendicular.

Solving for y, the equations become

$$y = \tfrac{3}{4}x + \tfrac{1}{4}, \quad y = -\tfrac{4}{3}x + \tfrac{5}{3}.$$

The two slopes are

$$m_1 = \tfrac{3}{4}, \quad m_2 = -\tfrac{4}{3}.$$

Since

$$m_2 = -\frac{1}{m_1}$$

the lines are perpendicular.

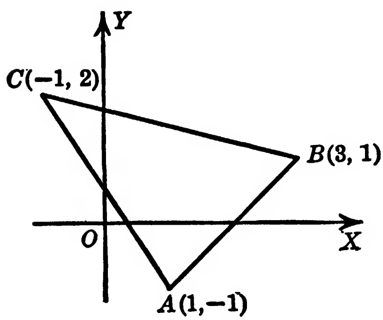

FIGURE 70.

Example 2. The vertices of a triangle are $A(1, -1)$, $B(3, 1)$, $C(-1, 2)$. Find the interior angle A.

If the angle A is considered positive, Figure 70 shows that it extends from AB to AC. The slope of AB is 1 and that of AC is $-\frac{3}{2}$. Hence

$$\tan A = \frac{-\frac{3}{2} - 1}{1 - \frac{3}{2}} = 5,$$

$$A = \tan^{-1} 5.$$

57. Tangent Line and Normal to a Curve. Let (x_1, y_1) be a fixed point on a curve, $\dfrac{dy}{dx}$ the derivative obtained from the equation of the curve, and

$$m_1 = \left(\frac{dy}{dx}\right)_1$$

the value this derivative takes when $x = x_1$, $y = y_1$.

The tangent to the curve at $P_1(x_1, y_1)$, being a line through P_1 with slope m_1, is represented by the equation

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1). \tag{1}$$

The line P_1N perpendicular to the tangent at its point of contact with the curve is called the *normal* to the curve at that point.

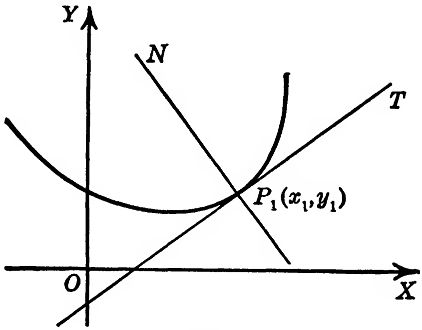

FIGURE 71.

Since the slope of the perpendicular is the negative reciprocal of the slope of the tangent (§56), the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{\left(\dfrac{dy}{dx}\right)_1}(x - x_1). \tag{2}$$

In equations (1) and (2) it should be noted that x_1, y_1 are coordinates of a point on the curve but that (x, y) is in general not on the curve.

Example. Find the equations of the tangent and normal to the curve $xy = 2$ at the point (1, 2).

From the equation $xy = 2$ we find by differentiation

$$x\frac{dy}{dx} + y = 0,$$

whence

$$\frac{dy}{dx} = -\frac{y}{x}.$$

At (1, 2) the slope of the tangent is

$$\left(\frac{dy}{dx}\right)_1 = -\frac{2}{1} = -2.$$

The equation of the tangent is therefore

$$y - 2 = -2(x - 1)$$

and that of the normal

$$y - 2 = \tfrac{1}{2}(x - 1).$$

58. Newton's Method. For a short distance the tangent to a smooth curve forms almost a continuation of the curve. This is the basis of a method due to Newton, for the approximate solution of equations.

Let

$$f(x) = 0$$

be the equation. A root of this equation is the abscissa of a point at which the curve

$$y = f(x)$$

meets the x-axis. By trial suppose $f(x_0)$ is found to be small. The tangent at x_0 is

$$y - f(x_0) = f'(x_0)(x - x_0). \quad (1)$$

This crosses the x-axis at

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \quad (2)$$

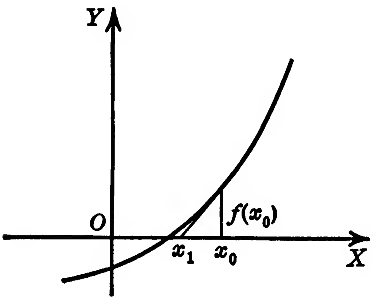

FIGURE 72.

If the curve and tangent are nearly coincident over the range (x_0, x_1) the value x_1 should be an approximate root of the equation.

If necessary the process may be repeated, giving

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \quad (3)$$

etc. Whether and how fast the process will converge depends on the function $f(x)$ and on the initial value x_0. Conditions favorable to convergence are evidently that $f(x_0)$ be small and $f'(x_0)$ large.

Example. Find an approximate solution of the equation

$$x^3 + 4x - 6 = 0.$$

Writing

$$f(x) = x^3 + 4x - 6$$

we find by trial

$$f(1.1) = -0.269.$$

By Newton's method we obtain

$$x_1 = 1.1 - \frac{f(1.1)}{f'(1.1)} = 1.1 + \frac{0.269}{7.63} = 1.1353.$$

The value correct to four decimals is 1.1347.

59. Angle between Two Curves. A point of intersection of two curves has coördinates that satisfy both equations. To find the intersections of two curves we thus solve their equations simultaneously.

By the angle between two curves at a point of intersection we mean the angle between their tangents at that point. If m_1 and

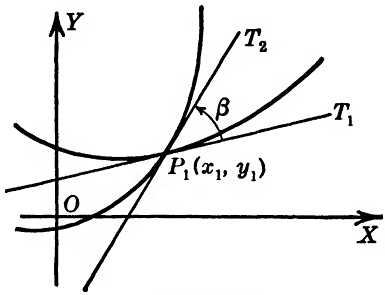

FIGURE 73.

m_2 are the slopes at a point of intersection and β is the angle from the curve with slope m_1 to that with slope m_2, by equation (5), §56,

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

Two curves are said to be *orthogonal* if they intersect at right angles. The condition for this is that either slope at a point of intersection be the negative reciprocal of the other.

Example 1. Find the angle between the line $y = x$ and the curve $y = x^2$ at each intersection.

Solving simultaneously, we find that the line and curve intersect at (1, 1) and (0, 0). The slope of the line is

$$m_1 = 1.$$

The slope of the curve at any point x is

$$\frac{dy}{dx} = 2x.$$

At (1, 1) the slope of the curve is 2 and the angle β_1 from line to curve is determined by

$$\tan \beta_1 = \frac{2 - 1}{1 + 2} = \frac{1}{3}.$$

At (0, 0) the slope of the curve is zero and

$$\tan \beta_2 = \frac{0 - 1}{1 + 0} = -1,$$

whence

$$\beta_2 = -45°.$$

The negative sign in this last case signifies that the acute angle is measured clockwise from the line to the curve.

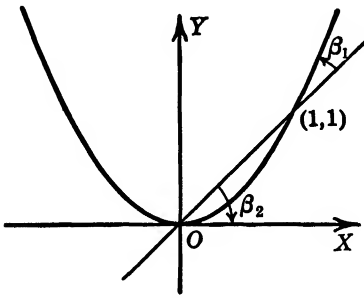

FIGURE 74.

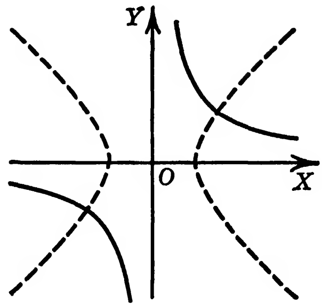

FIGURE 75.

Example 2. Show that if a and b are any constants the curves

$$x^2 - y^2 = a, \; xy = b$$

intersect orthogonally.

From the first equation we obtain

$$\frac{dy}{dx} = \frac{x}{y},$$

and from the second

$$\frac{dy}{dx} = -\frac{y}{x}.$$

At a point of intersection x and y have the same values in both equations. The second slope being the negative reciprocal of the first, the curves are orthogonal at all intersections.

60. Distance between Two Points. To determine the distance between two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ construct the triangle (Figure 76) with sides P_1R, RP_2 parallel respectively to OX and OY. The lengths of these sides are

$$P_1R = \pm(x_2 - x_1), \quad RP_2 = \pm(y_2 - y_1),$$

the algebraic signs depending on the positions of P_1 and P_2. Since the expression

$$P_1P_2 = \sqrt{\overline{P_1R}^2 + \overline{RP_2}^2}$$

contains only the squares of $\overline{P_1R}$ and $\overline{RP_2}$ it follows, however, in every case that

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1}$$

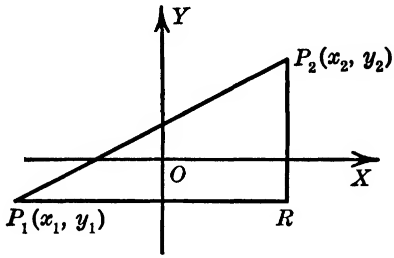

FIGURE 76.

Example. Find the point equidistant from the three points $A(9, 0)$, $B(-6, 3)$, $C(5, 6)$.

Let $P(x, y)$ be the point required. Since $PA = PB$ and $PA = PC$, we have

$$\sqrt{(x - 9)^2 + y^2} = \sqrt{(x + 6)^2 + (y - 3)^2},$$

$$\sqrt{(x - 9)^2 + y^2} = \sqrt{(x - 5)^2 + (y - 6)^2}.$$

Squaring and canceling, we obtain

$$5x - y = 6,$$

$$2x - 3y = 5.$$

By solving these last equations simultaneously, we find $x = 1$, $y = -1$. The required point is therefore $(1, -1)$.

61. Equation of a Circle. A circle is the locus of points at constant distance, equal to the radius, from a fixed point called the center.

Let $C(h, k)$ be the center of a circle (Figure 77) and r its radius. If $P(x, y)$ is any point on the circle,

$$r = CP = \sqrt{(x - h)^2 + (y - k)^2}.$$

Thus

$$(x - h)^2 + (y - k)^2 = r^2 \tag{1}$$

is the equation of the circle.

When expanded (1) becomes

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is an equation of the form

$$A(x^2 + y^2) + Bx + Cy + D = 0, \tag{2}$$

in which A, B, C, D are constants.

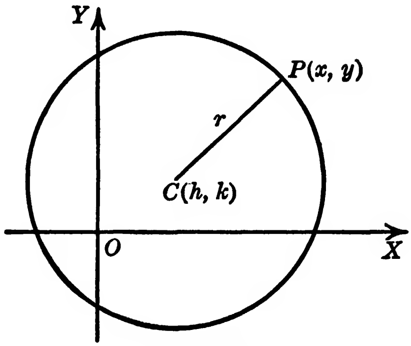

FIGURE 77.

Every circle is represented by an equation of the form (2). Conversely, if A is not zero any equation of this kind represents a circle if it represents any curve. To show this divide by A and complete the squares of terms containing x and those containing y separately. The result is

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \frac{B^2 + C^2 - 4AD}{4A^2}. \tag{3}$$

If the right side of this equation is positive, comparison with (1) shows that it represents a circle with center $\left(-\frac{B}{2A}, -\frac{C}{2A}\right)$ and radius

$$r = \sqrt{\frac{B^2 + C^2 - 4AD}{4A^2}}.$$

If the right side of (3) is zero, the radius is zero and the circle shrinks to a point. If the right side is negative, since the sum of the squares of real numbers is positive, there is no real locus.

The equation of a circle contains three constants, for example, h, k, r. These constants can in general be determined to satisfy

three equations and so the circle can be made to fulfill three conditions. Thus a circle can be passed through three points not on a line, can be made to touch three lines not parallel or through a point, etc.

Example 1. Find the center and radius of the circle

$$2x^2 + 2y^2 - 3x + 4y = 3.$$

Dividing by 2 and completing the squares, we have

$$(x - \tfrac{3}{4})^2 + (y + 1)^2 = \tfrac{49}{16}.$$

Comparison with (1) shows that the center of the circle is $(\frac{3}{4}, -1)$ and its radius is $\frac{7}{4}$.

Example 2. Find the equation of the circle which passes through the origin and through the two points in which the circle $x^2 + y^2 = 3$ intersects the straight line $x + y = 1$.

To do this we use the equation

$$x^2 + y^2 - 3 + k(x + y - 1) = 0, \tag{4}$$

k being constant. If $x^2 + y^2 - 3$ and $x + y - 1$ are both zero this equation is satisfied. Thus it represents a locus which passes through the intersections of the circle and line. Since (4) is of the form (2) this locus is a circle. To make the circle pass through the origin we place $x = 0$, $y = 0$, thus obtaining

$$-3 + k(-1) = 0$$

and $k = -3$. When k has this value equation (4) satisfies all the conditions of the problem. Thus

$$x^2 + y^2 - 3 - 3(x + y - 1) = 0,$$

or its equivalent

$$x^2 + y^2 - 3x - 3y = 0,$$

is the equation required.

62. Parabola. *A parabola is the locus of points equally distant from a fixed point and a fixed straight line.* The fixed point is called the *focus*, and the fixed line the *directrix*.

Take the x-axis through the focus and the origin midway between the focus and directrix. In Figure 78, F is the focus, RS the directrix, and

$$DO = OF = p. \tag{1}$$

The focus is thus the point $F(p, 0)$. If $P(x, y)$ is any point on the parabola and NP is perpendicular to RS, by definition

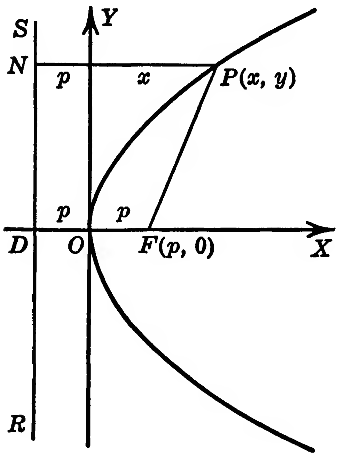

FIGURE 78.

$$FP = NP, \tag{2}$$

that is

$$\sqrt{(x - p)^2 + y^2} = x + p. \tag{3}$$

This is the equation of the parabola. By squaring and canceling it becomes

$$y^2 = 4px. \tag{4}$$

Lines, $x =$ constant, perpendicular to the x-axis cut the curve in pairs of points

$$y = \pm\sqrt{4px}$$

at equal distances above and below the x-axis. We express this by calling the x-axis an *axis of symmetry* of the curve. Since it is the only axis of symmetry, it is called the *axis* of the parabola. The point O where the axis intersects the curve is called the *vertex* of the parabola.

In equation (4), y is the perpendicular from a point on the curve to the axis, and x is the distance along the axis from this perpendicular to the vertex. If the vertex is not the origin but some point $A(h, k)$, Figure 79, these distances are

$$MP = y - k, \quad AM = x - h$$

and so the equation of the parabola is

$$(y - k)^2 = 4p(x - h). \tag{5}$$

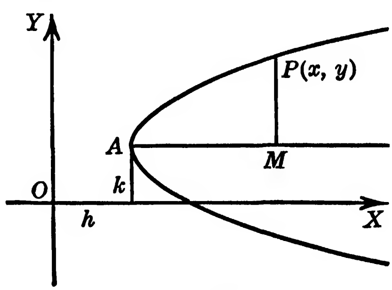

FIGURE 79.

If the axis of the parabola is parallel to the y-axis, the variables x and y are interchanged giving

$$(x - h)^2 = 4p(y - k) \tag{6}$$

as the equation of the curve.

Equations (5) and (6) may be written

$$Ay^2 + Bx + Cy + D = 0, \tag{7}$$

$$Ax^2 + Cx + By + D = 0, \tag{8}$$

A, B, C, D being constants. Conversely, by completing the squares it can easily be shown that any equation of the form (7) or (8) represents a parabola if A and B are not zero.

Since division by a constant has no effect, equations (7) and (8) contain three essential constants. A parabola with axis parallel to a coördinate axis can thus be made to satisfy three conditions, for example, to pass through three points not on a line.

When rotated about its axis a parabola generates a surface called a paraboloid of revolution. A mirror of this form has the property that rays of light parallel to the axis, striking its inner surface, are reflected to the focus, and rays issuing from the focus are reflected parallel to the axis. Such a mirror is consequently used in a reflecting telescope to focus light from a distant object, and in an automobile headlight to reflect light from a lamp near the focus in a substantially parallel beam.

When carrying a load proportional to the horizontal distance, the cable of a suspension bridge hangs in a parabola, and a parabolic arch experiences only tangential compression.

If air friction is neglected a projectile describes a parabola with vertical axis.

Example 1. Find the vertex, focus, and directrix of the parabola

$$y^2 - 2y - 2x = 5.$$

Completing the square of terms in y, this becomes

$$(y - 1)^2 = 2(x + 3).$$

Comparison with (5) shows that this is a parabola with vertex $(-3, 1)$, and that $4p = 2$. The axis being parallel to OX, the focus is at distance $p = \frac{1}{2}$ to the right of the vertex and the directrix at the same distance on the left. Thus the focus is $(-\frac{5}{2}, 1)$ and the directrix is $x = -\frac{7}{2}$.

Example 2. An arch has the form of a parabola with vertical axis. If the arch is 10 feet high at the center and 24 feet wide at the base, find the width QP (Figure 80) at a height 6 feet above the base.

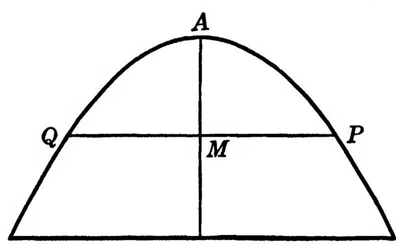

FIGURE 80.

If P is any point on the parabola, by equation (4)

$$\overline{MP}^2 = 4p\cdot AM.$$

Thus

$$\overline{PQ}^2 = 4\overline{MP}^2 = k\cdot AM$$

where $k = 16p$ is constant. At the base $\overline{QP} = 24$, $AM = 10$, and this becomes

$$\overline{24}^2 = k\cdot 10,$$

whence by division

$$\frac{\overline{PQ}^2}{24^2} = \frac{AM}{10}.$$

At height 6 feet, $AM = 4$ and

$$PQ = 24\sqrt{\tfrac{4}{10}} = 15.18 \text{ ft.}$$

Example 3. Parabolas with the same axis and focus are called *confocal.* The parabolas with focus at the origin and axis OX are obtained by assigning values to p in the equation

$$y^2 = 4px + 4p^2. \tag{9}$$

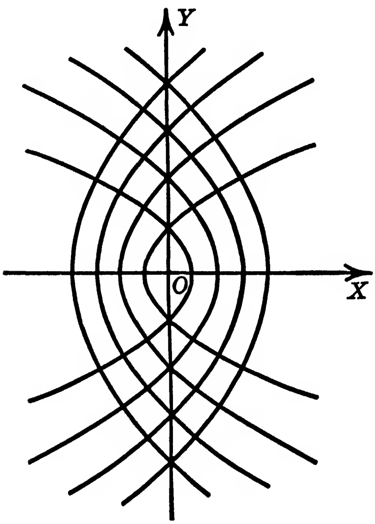

FIGURE 81.

Show that two of these curves pass through any point, not on the axis, and that these intersect at right angles. We express this by saying the system of confocal parabolas is self-orthogonal.

The two values obtained by solving (9) for p determine two parabolas through the point (x, y). Differentiation gives

$$m = \frac{dy}{dx} = \frac{2p}{y} \tag{10}$$

for the slope of either curve at (x, y). Eliminating p from (9) and (10) we obtain

$$m^2 + \frac{2mx}{y} - 1 = 0$$

as an equation the two slopes must satisfy. If m_1 and m_2 are the two roots, this equation can be written

$$(m - m_1)(m - m_2) = 0.$$

Comparison with the preceding equation indicates that

$$m_1m_2 = -1,$$

and this shows that the two curves are orthogonal [§56, (3)].

63. Ellipse. *An ellipse is the locus of points the sum of whose distances from two fixed points is constant.* The two fixed points are called *foci* of the ellipse.

Let F', F be the foci, take the x-axis through these points, the

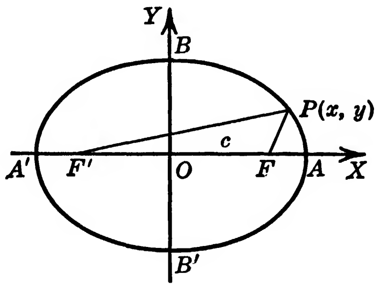

FIGURE 82.

origin midway between them, and let $2c$ be the distance $F'F$. The foci are then

$$F'(-c, 0),\ F(c, 0).$$

If $P(x, y)$ is any point on the ellipse and $2a$ is the sum of its distances from the foci, we have

$$F'P + FP = 2a, \tag{1}$$

that is

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a, \tag{2}$$

as the equation of the curve.

Transposing one of the square roots, squaring both sides, and solving for the remaining square root, we find

$$\begin{aligned} FP &= \sqrt{(x - c)^2 + y^2} = a - \frac{c}{a}x, \\ F'P &= \sqrt{(x + c)^2 + y^2} = a + \frac{c}{a}x. \end{aligned} \tag{3}$$

By clearing radicals either of these equations gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{4}$$

In the triangle $F'FP$ the sum of two sides, $F'P + FP = 2a$, is greater than the third side, $F'F = 2c$. Thus a is greater than c,

and $a^2 - c^2$ is a positive quantity which we denote by b^2. Substituting

$$a^2 - c^2 = b^2, \tag{5}$$

the equation of the ellipse takes the final form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{6}$$

A line $x =$ constant or $y =$ constant perpendicular to either coördinate axis cuts the curve in pairs of points

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2},$$

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2},$$

on opposite sides and equally distant from the axis. The curve is thus symmetrical with respect to OX and OY, which are therefore called axes of the curve. Their intersection O is called the center of the ellipse.

When $y = 0$ these equations give $x = \pm a$, and when $x = 0$, $y = \pm b$. The numbers a, b are therefore the distances OA, OB along the axes from the center to the curve. The lines $A'A$, $B'B$ are called the major and minor axis, and a and b are called semiaxes. The intersections A', A of the curve and major axis are called vertices.

From the definition of the curve, or from equation (5), it is seen that the distance from the end of the minor axis to either focus is equal to the major semiaxis. When a graph of the ellipse is given the foci can therefore be located by describing a circle of radius a (assuming that $A'A$ is the major axis) and center B and determining its intersections with the major axis.

The ratio

$$e = \frac{OF}{OA} = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} \tag{7}$$

is called the *eccentricity* of the ellipse. It is always less than unity. If it is zero, $a = b$ and the ellipse becomes a circle. As e increases, the ratio of major to minor axis increases and the ellipse becomes more and more eccentric.

Equations (3) can be written

$$FP = a - ex = e\left(\frac{a}{e} - x\right) = e \cdot NP,$$

$$F'P = a + ex = e\left(\frac{a}{e} + x\right) = e \cdot N'P.$$

The lines

$$x = \pm \frac{a}{e} \tag{8}$$

are called *directrices* of the ellipse. In Figure 83 they are the lines RS, $R'S'$. The above equations show that

$$\frac{FP}{NP} = \frac{F'P}{N'P} = e. \tag{9}$$

Thus, *an ellipse is the locus of a point the ratio of whose distance from a fixed point* (a focus) *to its distance from a fixed line* (the corresponding directrix) *is equal to a constant less than* 1 (the eccentricity).

In the above discussions it is assumed that the major axis is $A'A$. If the major axis is $B'B$, appropriate changes must be made in expressions for foci, eccentricity, and directrices.

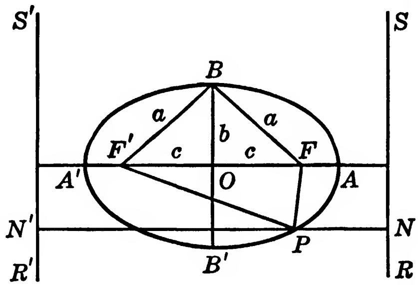

FIGURE 83.

In equation (6) x and y are the displacements from the axes of the curve to the point $P(x, y)$. If the center of the curve, instead of being the origin, is the point (h, k), these displacements are $x - h$, $y - k$, and the equation of the ellipse becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{10}$$

If a circle is deformed in such a way that the distances of its points from a fixed diameter are all changed in the same ratio, the resulting curve is an ellipse.

When rotated about its major axis an ellipse generates an ellipsoid of revolution. If the inner surface of such an ellipsoid is silvered, light issuing from one focus is reflected to the other focus. If the exterior surface is silvered, light converging toward one focus is reflected as if it came from the other focus.

Under the law of gravitation the orbit of a planet moving around the sun is an ellipse with the sun at one of its foci.

Example 1. Show that

$$25x^2 + 9y^2 + 50x - 18y = 191$$

represents an ellipse, and find its center, axes, foci, eccentricity, and directrices.

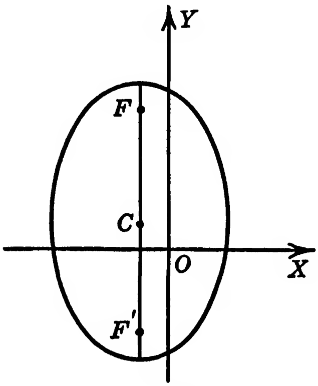

FIGURE 84.

The equation can be written

$$25(x^2 + 2x) + 9(y^2 - 2y) = 191.$$

Completing the squares in the parentheses, this becomes

$$25(x + 1)^2 + 9(y - 1)^2 = 225,$$

or

$$\frac{(x + 1)^2}{9} + \frac{(y - 1)^2}{25} = 1.$$

Comparison with (10) shows that this represents an ellipse with center $(-1, 1)$, axes $x = -1$, $y = 1$, and semiaxes $a = 3$, $b = 5$. Since b is larger than a the foci are on the vertical axis at distances

$$c = \sqrt{b^2 - a^2} = 4$$

above and below the center. Thus they are $F'(-1, -3)$, $F(-1, 5)$. The eccentricity is

$$e = \frac{\sqrt{b^2 - a^2}}{b} = \frac{4}{5},$$

and the directrices are horizontal at distances

$$\frac{b}{e} = \frac{25}{4}$$

above and below the center. Consequently, their equations are

$$y = \tfrac{29}{4}, \quad y = -\tfrac{21}{4}.$$

Example 2. Show that if $P_1(x_1, y_1)$ is on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the equation of the tangent at P_1 is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \tag{11}$$

If P_1 is not on the ellipse determine what line the last equation represents.

From the equation of the curve we obtain

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

If $P_1(x_1, y_1)$ is on the ellipse, the tangent at P_1 is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1).$$

Simplifying and using the equation

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1,$$

this can be reduced to the form (11).

If P_1 is not on the ellipse, let $P'(x', y')$ and $P''(x'', y'')$ be the points in which the line

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

and ellipse intersect. Since P' is on the line,

$$\frac{x'x_1}{a^2} + \frac{y'y_1}{b^2} = 1.$$

This, however, states that the tangent at P', namely

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1,$$

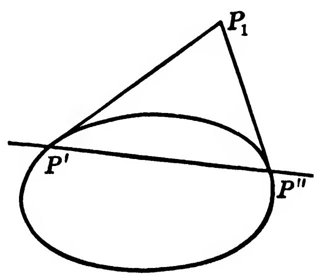

FIGURE 85.

passes through P_1. In a similar way we show that the tangent at P'' also passes through P_1. Thus (11) represents the line $P'P''$ through the points of contact of tangents which pass through P_1 (Figure 85).

64. Hyperbola. *The locus of a point, the difference of whose distances from two fixed points is constant, is called a hyperbola.* The two fixed points are called *foci.*

Let F', F be the foci; take the x-axis through these points and the origin midway between them; and let $2c$ be the distance $F'F$. The foci are then

$$F'(-c, 0),\ F(c, 0).$$

If $P(x, y)$ is any point on the hyperbola and $2a$ the difference of its distances from the foci,

$$F'P - FP = \pm 2a, \tag{1}$$

that is,

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \tag{2}$$

is the equation of the curve.

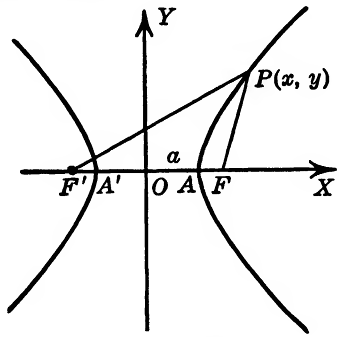

FIGURE 86.

Transposing one of the square roots, squaring, and solving for the other square root, we find

$$\begin{aligned} FP &= \sqrt{(x-c)^2 + y^2} = \mp\left(a - \frac{c}{a}x\right), \\ F'P &= \sqrt{(x+c)^2 + y^2} = \pm\left(a + \frac{c}{a}x\right). \end{aligned} \tag{3}$$

Clearing radicals either of these equations gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \tag{4}$$

as in the case of the ellipse.

In the triangle $F'FP$, the difference of two sides $2a$ is less than the third side $F'F = 2c$. Thus a is less than c and $a^2 - c^2$ is a negative quantity which we denote by $-b^2$. Substituting

$$c^2 - a^2 = b^2, \tag{5}$$

the equation of the hyperbola takes the final form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{6}$$

Since the equation contains only even powers of x and y the hyperbola is symmetrical with respect to both coördinate axes. They are therefore called the axes of the curve, and their intersection is called the center. The distances a and b are called semiaxes.

When $y = 0$, $x = \pm a$; but when $x = 0$, y is imaginary. Thus the axis through the foci, called the *transverse* axis, intersects the curve in two points A', A, called *vertices*, at distance a on each side of the center, but the other axis, called the *conjugate* axis, does not intersect the curve. The hyperbola thus consists of two separate parts, namely two symmetrical branches, on opposite sides of the conjugate axis.

The ratio

$$e = \frac{OF}{OA} = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} \tag{7}$$

is called the *eccentricity* of the hyperbola. It is always greater than unity. As the *eccentricity* decreases to unity the two branches of the curve draw closer to the transverse axis. As the eccentricity increases, the two branches tend toward lines perpendicular to the transverse axis.

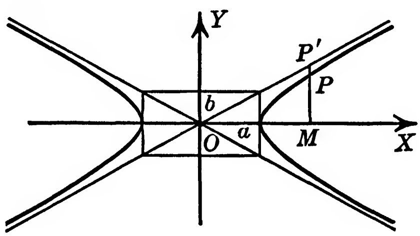

FIGURE 87.

Solved for y the equation of the hyperbola becomes

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}.$$

Assuming P in the first quadrant, the vertical distance between P and the line $y = \frac{b}{a}x$ is

$$PP' = \frac{b}{a}x - y = \frac{b}{a}[x - \sqrt{x^2 - a^2}] = \frac{ab}{[x + \sqrt{x^2 - a^2}]}.$$

As x increases, this tends to zero. Similarly, if $P(x, y)$ is a point of the curve in any quadrant, the distance between P and one of the lines

$$y = \pm \frac{b}{a} x \tag{8}$$

tends to zero as $|x|$ tends to infinity. These lines are called *asymptotes* of the curve.

If a and b are equal the equation of the hyperbola becomes

$$x^2 - y^2 = a^2. \tag{9}$$

The asymptotes

$$y = \pm \frac{b}{a} x = \pm x$$

are then perpendicular to each other. Such a hyperbola is called *equilateral* or *rectangular*.

If the center of the hyperbola, instead of being the origin, is the point (h, k), and the transverse axis is parallel to OX, the equation of the curve is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \tag{10}$$

If the transverse axis is parallel to OY this is replaced by

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \tag{11}$$

and appropriate changes must be made in the expressions for eccentricity, vertices, etc.

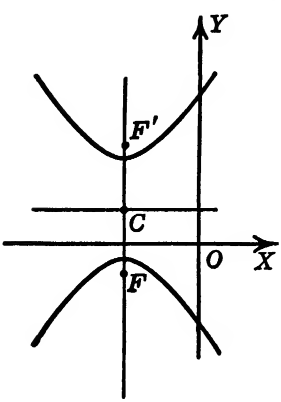

FIGURE 88.

Example. Show that

$$y^2 - 2x^2 - 2y - 8x - 9 = 0$$

represents a hyperbola, and find its center, eccentricity, and foci.

Completing the squares this equation becomes

$$(y^2 - 2y + 1) - 2(x^2 + 4x + 4) = 2,$$

which is equivalent to

$$\frac{(y-1)^2}{2} - \frac{(x+2)^2}{1} = 1.$$

This represents a hyperbola with center $(-2, 1)$, semiaxes

$$a = 1, \quad b = \sqrt{2},$$

and transverse axis parallel to OY. In the expression for eccentricity a and b must be interchanged, giving

$$e = \frac{\sqrt{a^2 + b^2}}{b} = \sqrt{\frac{3}{2}}.$$

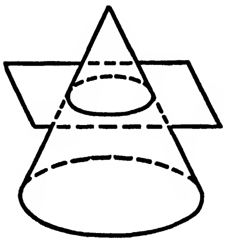

FIGURE 89.

The distance from the center to the foci is

$$c = \sqrt{a^2 + b^2} = \sqrt{3}.$$

The foci, being on a line parallel to the y-axis at distance c above and below the center, are

$$(-2, 1 \pm \sqrt{3}).$$

65. Conic Sections. The circle, ellipse, parabola, and hyperbola are often called *conic sections*. The reason for this is that all can be obtained as sections cut from a right circular cone by a plane.

If the cutting plane is parallel to the base of the cone (Figure 89) the section is obviously a circle.

If the plane is inclined to the base of the cone but at an angle less than that between the base and generators (Figure 90) the section is an ellipse. To show this we construct two spheres touching the cone in circles AB, $A'B'$ and the plane in points F, F'. If P is any point on the plane and cone and Q, Q' are the points in which the generator through P cuts the two circles, the lines PF, PQ, PF', PQ' are tangent to the two spheres. Since all tangents from a given point to a given sphere are equal in length,

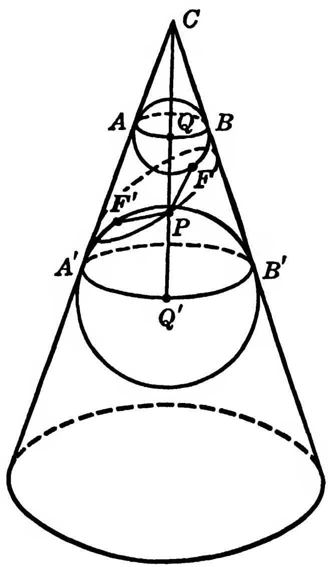

FIGURE 90.

$$PF = PQ, \quad PF' = PQ'$$

and so

$$PF + PF' = PQ + PQ' = QQ'$$

is the distance along the generator between the two circles. Since this distance has the same value for all points P on the section, this proves that the section is an ellipse and that F, F' are its foci.

If the plane has the same inclination to the base of the cone as the generators (Figure 91), the section is a parabola. There is

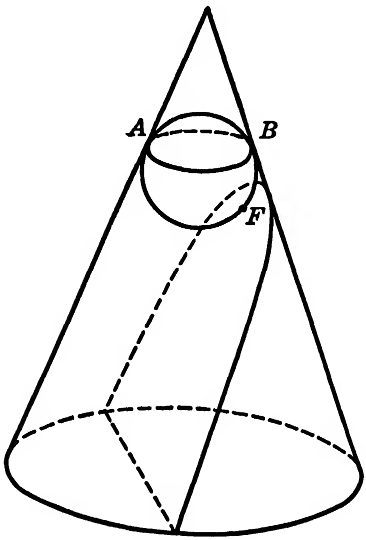

FIGURE 91.

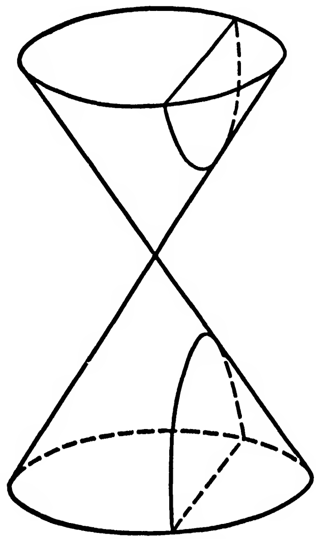

FIGURE 92.

then a single sphere tangent to the cone along a circle and tangent to the plane in a point. This point is the focus of the parabola, and the directrix is the line in which the plane of the circle cuts the plane of the parabola.

Finally, if the plane is inclined to the base of the cone at an angle greater than that between the base and generators, the plane will also cut the cone formed by the extension of these generators (Figure 92). The section consisting thus of two parts is a hyperbola. By a discussion similar to that presented for the ellipse it may be shown that the foci of the hyperbola are the points where the plane touches the two spheres tangent to the cone and plane.

66. Transformation of Coördinates. It is sometimes desirable to move the coördinate axes to a new position. The coördinates of a given point are changed, and the equations connecting old and new coördinates are needed. There are two simple cases a combination of which gives any motion. These are *translation*, in which the origin is moved to a new position without changing the direction of the axes, and *rotation*, in which the origin is left fixed and the axes turned like a rigid frame about it.

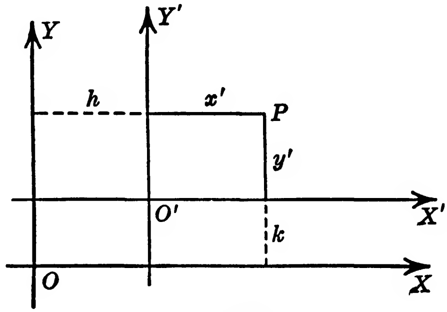

FIGURE 93.

(1) *Translation of the Axes.* Let OX, OY be the axes in their first position, $O'X'$, $O'Y'$ the axes after motion, and h, k the coördinates of the new origin with respect to the old axes. If x, y are the coördinates of a point P with respect to the old axes, and x', y' the coördinates of the same point with respect to the new, from Figure 93 it is seen at once that

$$x = x' + h, \quad y = y' + k \tag{1}$$

are the equations of transformation.

(2) *Rotation of the Axes.* Let OX, OY be the original axes and OX', OY' the new axes into which they are changed by rotation through the angle ϕ about the origin. Let x, y be the coördinates of a point P with respect to the original axes, and x', y' the coördinates of the same point with respect to the new. In Figure 94, x and y are the horizontal and vertical displacements in moving from O to P, and these are the sums of the corresponding displacements in moving from O to M' and from M' to P.

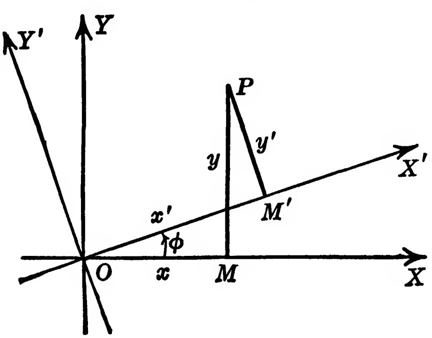

FIGURE 94.

Thus

$$x = x' \cos \phi - y' \sin \phi,$$
$$y = x' \sin \phi + y' \cos \phi, \tag{2}$$

the algebraic sign of each displacement being determined by its direction parallel to the corresponding axis.

Example 1. By translation of the axes reduce the equation

$$x^2 + 2y^2 - 6x + 4y = 7$$

to a form that has no terms of first degree in the new coördinates.

Substituting

$$x = x' + h, \quad y = y' + k$$

the equation becomes

$$(x')^2 + 2(y')^2 + (2h - 6)x' + (4k + 4)y' + h^2 + 2k^2 - 6h + 4k - 7 = 0.$$

The coefficients of x' and y' will be zero if

$$2h - 6 = 4k + 4 = 0.$$

Thus $h = 3$, $k = -1$ are the coördinates of the new origin with respect to the old axes. Substituting these values, the equation of the curve becomes

$$(x')^2 + 2(y')^2 - 18 = 0.$$

Example 2. Determine the new equation into which

$$xy = k$$

is changed when the axes are rotated 45°.

The equations for rotation through $\phi = 45°$ are

$$x = x' \cos \phi - y' \sin \phi = \frac{x' - y'}{\sqrt{2}}.$$

$$y = x' \sin \phi + y' \cos \phi = \frac{x' + y'}{\sqrt{2}}.$$

Substituting these values, the above equation becomes

$$(x')^2 - (y')^2 = 2k.$$

Thus $xy = k$ is the equation of a rectangular hyperbola with the coördinate axes as asymptotes.

67. Invariants. A transformation from one set of rectangular coördinates to another is called *orthogonal.* With respect to such changes geometrical quantities are of two kinds, those dependent on the position of the axes and those independent of the axes. Quantities of the second kind are called *invariant,* or, more accurately, invariant under orthogonal linear transformations. For example, the radius of a circle is invariant (does not depend on

the position of the axes), but the coördinates of its center are not invariant (change when the axes change position).

The equation of a curve is not invariant, but there are some things connected with it which are invariant. One of the simplest is the degree of an equation in rectangular coördinates. Wherever the axes are placed this degree is always the same. To see this it is sufficient to note that the equations for any change of rectangular coördinates have the form

$$x = a_1x' + a_2y' + a_3, \quad y = b_1x' + b_2y' + b_3,$$

$a_1, a_2, a_3, b_1, b_2, b_3$ being constants. By such a substitution a term x^my^n is replaced by a sum of terms no one of which is of higher than the $(m + n)$th degree in x' and y'. Hence by change of rectangular axes the degree of an equation cannot be increased. Neither can it be diminished; for, then, changing back to the old axes would increase it. Therefore the degree of an equation in rectangular coördinates is invariant.

If an equation in x and y can be factored, so can the new equation resulting through a change of axes, for each factor of the old equation will be changed into a factor of the new. Since real expressions transform into real, if either equation has real factors, the other does also.

68. General Equation of the Second Degree. An equation of the second degree in rectangular coördinates has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

A, B, C, D, E, F being constants. To determine the locus represented we simplify the equation as much as possible by change of coördinate axes.

First, by rotating the axes through a proper angle we make the coefficient of xy zero. Then, by translating to parallel axes, as in Example 1, §66, we reduce the coefficient of a first power to zero if the second power of the same coördinate occurs, and we reduce the constant term to zero if only a first power occurs. By proper choice of the axes any equation of the second degree in rectangular coördinates can thus be reduced to one of the following forms:

$$ax^2 + by^2 + c = 0, \tag{2}$$

$$y^2 + cx = 0, \tag{3}$$

a, b, c being constants.

If c is zero, equation (2) or (3) is a product of first-degree factors. If c is not zero, the locus is immediately seen to be a circle, ellipse, parabola, hyperbola, or entirely imaginary. The last case occurs when a, b, c in (2) have the same algebraic sign. We thus conclude that,

If a second-degree equation in rectangular coördinates is not a product of first-degree factors, it represents a circle, ellipse, parabola, hyperbola, or an imaginary locus.

To determine which of these loci a particular second-degree equation represents we note that in a rotation of axes the second degree part of equation (1), namely

$$Ax^2 + Bxy + Cy^2,$$

transforms into the second-degree part of the new equation. Also the second-degree part

$$\frac{x^2}{a^2} - \frac{y^2}{b^2}$$

in the standard equation of the hyperbola is the product of real first-degree factors

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right),$$

whereas the second-degree part of the equation of a parabola is a complete square, and that of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is not factorable into real first-degree factors. Thus we conclude that if equation (1) represents a *real curve* that curve is a hyperbola, parabola, or ellipse according as

$$Ax^2 + Bxy + Cy^2$$

is the product of real and distinct first-degree factors, is a complete square, or is not factorable into real first-degree factors.

69. Graphs of Algebraic Equations. The equation of a curve being given, any number of points on the curve can be determined by assigning values to either coördinate, calculating the corresponding values of the other coördinate, and plotting the resulting points. When enough points have been located a smooth curve drawn through them may be taken as an approximation to the

required curve. We desire as quickly as possible to obtain a satisfactory approximation. To some extent this is accomplished by plotting points rather sparsely where the curve is nearly straight and more closely where it bends rapidly. The following are features it may be helpful to note:

(1) Points where the curve crosses the axes and, if possible, the points between crossings at which it reaches a maximum distance from the axis.

(2) Values of either coördinate for which the other coördinate is real and values for which it is imaginary.

(3) Symmetry.

(4) Infinite values of the coördinates, asymptotes.

(5) Direction of the curve near a point.

(1) *Intersections with the Axes.* The points where a curve meets the x-axis are found by letting $y = 0$ in the equation and solving

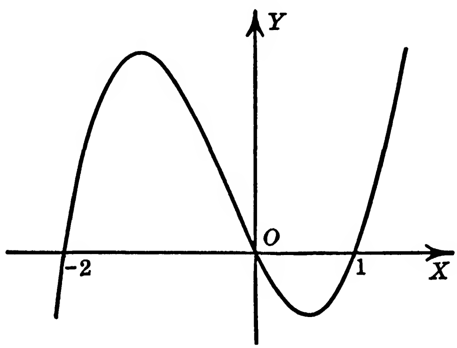

FIGURE 95.

for x. Similarly, points on the y-axis are found by letting $x = 0$ and solving for y. The x-coördinates of points on the x-axis and the y-coördinates of those on the y-axis are sometimes called *intercepts* of the curve on the coördinate axes.

Example 1. $y = x^3 + x^2 - 2x$.
Factoring the right side results in

$$y = x(x - 1)(x + 2).$$

The curve crosses the x-axis at $x = -2, 0, 1$. These values divide the x-axis and the curve into four parts. On the left of $x = -2$ all three factors of the product are negative and y is negative. Similarly y is found

to be negative between $x = 0$ and $x = 1$ and positive in the other two sections. By solving the equation

$$\frac{dy}{dx} = 3x^2 + 2x - 2 = 0$$

and substituting in the value of y, maxima and minima of y are found at $(-1.21, 2.11)$ and $(0.55, -0.63)$. Using these values and plotting a few additional points the curve in Figure 95 is obtained.

Example 2. $y - 2 = x^2(x - 3)$.

Here intersections with $y = 2$ should evidently be used. These occur at $x = 0$ and $x = 3$. The ordinate has a maximum at $(0, 2)$ and a minimum at $(2, -2)$. The curve is shown in Figure 96.

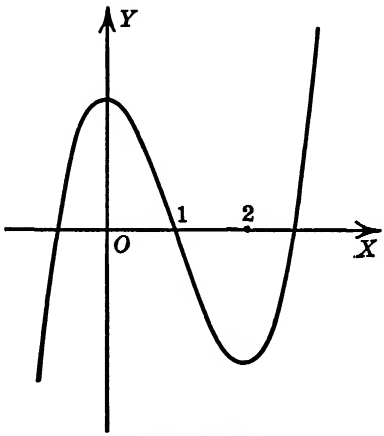

FIGURE 96.

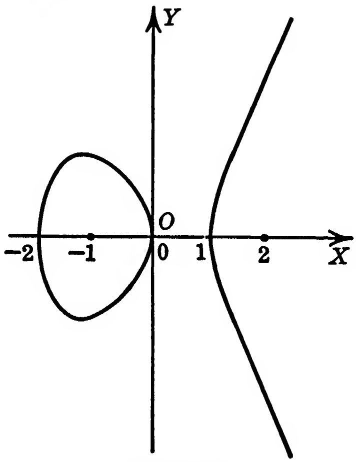

FIGURE 97.

(2) *Real and Imaginary Values.* When certain values are substituted for one coördinate the other coördinate may be real and when other values are substituted it may be imaginary. The plane is thus divided into strips in which there is a part of the curve and strips in which there is none. These being determined, the part of the curve in each strip may be plotted separately.

Example. $y^2 = x(x - 1)(x + 2)$.

The curve crosses the x-axis at its intersections with the lines $x = -2$, $x = 0$, $x = 1$. These lines divide the plane into four strips. On the left of $x = -2$, the product $x(x - 1)\ (x + 2)$ is negative and y is imaginary. Between $x = -2$ and $x = 0$, the product is positive and y is real. Similarly, between $x = 0$ and $x = 1$, y is imaginary, and on the right of $x = 1$, y is again real. The curve thus consists of two pieces, one between $x = -2$ and $x = 0$, the other on the right of $x = 1$ (Figure 97).

(3) *Symmetry.* Two points P, P' are said to be *symmetrical with respect to a line* if the segment PP' is bisected perpendicularly by that line. In Figure 98, for example, P and P' are symmetrical with respect to the x-axis, P and P'' with respect to the y-axis.

Two points P and P''' are called *symmetrical with respect to a point* O if the segment PP''' is bisected by O.

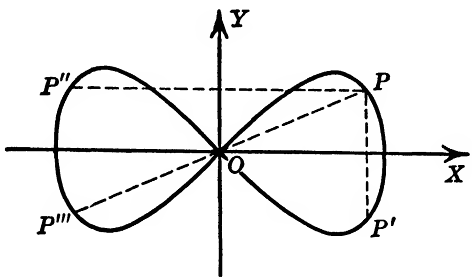

FIGURE 98.

A curve is called *symmetrical with respect to an axis* if it is generated by pairs of points symmetrical with respect to the axis. A curve is called *symmetrical with respect to a center* if it is generated by pairs of points symmetrical with respect to the center.

A curve $f(x, y) = 0$ is symmetrical with respect to the x-axis if

$$f(x, y) = f(x, -y).$$

For then it is generated by pairs of points (x, y), $(x, -y)$ symmetrical with respect to OX. In particular, *a curve is symmetrical with respect to the x-axis if its equation contains only even powers of y.*

Similarly, a curve is symmetrical with respect to the y-axis if its equation contains only even powers of x, and symmetrical with respect to the origin if all terms in its equation are of even degree or all are of odd degree.

Example. $y^2 = x^2 - x^4$.

The graph of this equation is shown in Figure 98. Since the equation contains only even powers of x and only even powers of y, the curve is symmetrical with respect to both axes and with respect to the origin.

(4) *Infinite Values.* One coördinate may increase without limit when the other approaches a certain value. In such a case the graph can be traced only until it runs off the paper. It thus consists of two or more parts not connected together. Sometimes

two adjacent parts go off in the same direction, as at A, B in Figure 99; sometimes they go in opposite directions, as at C, D; and sometimes there is no immediately adjacent part.

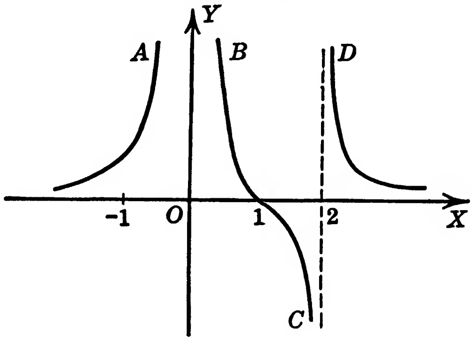

FIGURE 99.

If a branch of a curve when indefinitely prolonged approaches a straight line in such a way that the distance between the two tends to zero as limit, the line is called an *asymptote* of the curve.

Example. $y = \dfrac{x - 1}{x^2(x - 2)}$.

The graph is shown in Figure 99. As x tends to 0 or 2, y becomes infinite. When x is a little less than zero or a little greater than zero, y is large and positive. As x increases through zero the curve thus goes up one side of the y-axis and comes down the other. When x is a little less than 2, y is negative, and when x is a little greater than 2, y is positive. As x increases through 2 the curve thus goes down the left side of CD and then reappears at the top. When x increases without limit, y tends to zero. The two coördinate axes and the line CD are asymptotes of the curve.

(5) *Direction Near a Point.* To determine the shape of a curve near a point it is often helpful to find the direction along which the curve or a branch of the curve approaches that point.

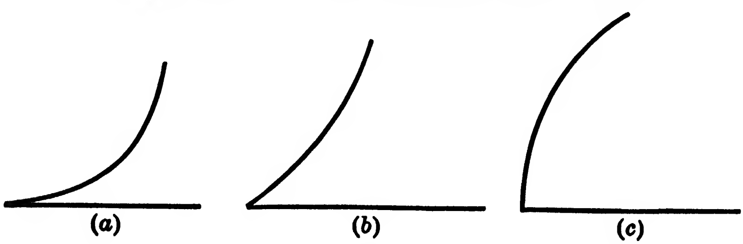

FIGURE 100.

In Figure 100, for example, are shown three ways in which a curve may approach a horizontal line. These are distinguished by the fact that the slope is zero in (*a*), is finite and different from zero in (*b*), and is infinite in (*c*).

Example 1. $y^2 = x^3$. The curve meets the x-axis at the origin but does not extend to the left of the y-axis. Since $y = x^{\frac{3}{2}}$, the slope

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$

is zero when $x = 0$. The curve thus has a sharp point, or cusp, at the origin as shown in Figure 101.

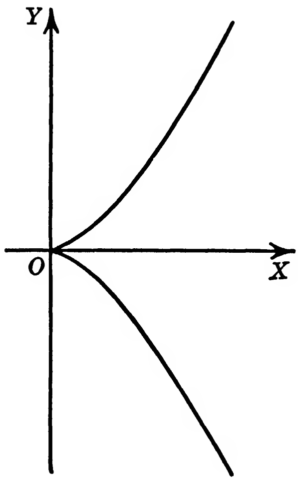

FIGURE 101.

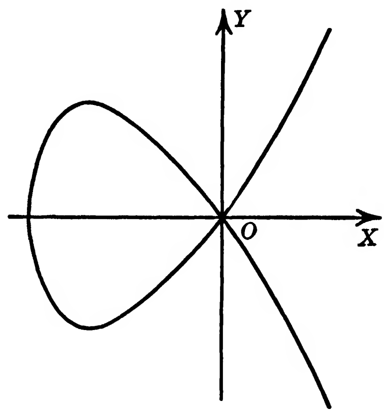

FIGURE 102.

Example 2. $y^2 = x^2(x + 2)$.

The curve is symmetrical with respect to the x-axis and passes through the origin, the slope at that point being

$$\frac{dy}{dx} = \pm\sqrt{2}.$$

At the origin the curve is tangent to the lines having these slopes.

PROBLEMS

1. Find the coördinates of the point $P(x, y)$ midway between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

2. On the line through $P_1(1, -3)$, $P_2(4, 6)$ find the point one-third of the way from P_1 to P_2.

3. On the line through $P_1(-2, 3)$, $P_2(2, -1)$ find the coördinates of the point $P(x, y)$ such that

$$PP_2 = -\tfrac{1}{2}P_1P_2.$$

4. Given $P_1(2, -3)$, $P_2(-1, 2)$, find the point on P_1P_2 which is twice as far from P_1 as from P_2. Also find the point on P_1P_2 produced which is twice as far from P_1 as from P_2.

5. On the line through $P_1(3, -4)$, $P_2(5, 2)$ find two positions of the point $P(x, y)$ such that the distance from P_1 to P is three times the distance from P_1 to P_2.

6. On the line through $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ find the point $P(x, y)$ such that

$$\frac{P_1P}{PP_2} = \frac{r_1}{r_2}.$$

7. On the line through $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ find the point $P(x, y)$ such that

$$P_1P = rP_1P_2.$$

8. Given the three points $A(-3, 3)$, $B(3, 1)$, $C(6, 0)$ on a line, find the fourth point $D(x, y)$ such that $AD : DC = -AB : BC$.

9. Find the equation of the straight line through $(2, -3)$ with slope 4.

10. Find the equation of the straight line through $(1, 2)$ parallel to the x-axis.

11. Find the equation of the straight line through $(3, -4)$ parallel to the y-axis.

12. Find the equation of the straight line through $(2, -1)$ and $(3, 2)$.

13. Find the equation of the straight line through $(2, 3)$ and $(2, -4)$.

14. Show that

$$\frac{x}{a} + \frac{y}{b} = 1$$

is the equation of a straight line with intercept a on the x-axis and b on the y-axis.

15. If the slopes of AB and BC are equal, show that the points A, B, C lie on a line. Show that $A(4, 1)$, $B(1, 2)$, $C(-5, 4)$ are on a line.

16. Show that the locus represented by

$$(x - 3y + 4)(2x + 4y - 1) = 0$$

consists of two straight lines.

17. Show that the line through $(1, -2)$ and $(-1, 4)$ is parallel to that through $(3, -5)$ and $(0, 4)$.

18. Find the equation of the straight line through $(-1, 2)$ parallel to the line $3x + 4y - 7 = 0$.

19. Find the equation of the straight line through $(2, 3)$ parallel to the line through $(-3, 1)$ and $(1, -4)$.

20. Show that the straight lines determined by the pairs of points $(2, 3)$, $(3, 5)$, and $(1, 7)$, $(3, 6)$ are perpendicular to each other.

21. Find the equation of the line through $(4, -1)$ perpendicular to the line $3x + y - 2 = 0$.

22. Find the equation of the line through $(-2, 3)$ perpendicular to the line through $(1, -2)$ and $(3, 4)$.

23. Find the angle from the positive direction of the x-axis to the line $y + x\sqrt{3} - 2 = 0$.

24. The sides of a triangle have the slopes $\frac{1}{2}$, 1, 2. Show that the triangle is isosceles.

25. Find the tangents of the interior angles of the triangle with vertices $A(1, 1)$, $B(3, 3)$, $C(2, 4)$.

26. The lines $x + y = 1$, $2y - x = 2$, and $3x - y = 3$ form a triangle. Find the tangents of its interior angles.

27. By showing that the angles ACB and ADB are equal prove that the points $A(2, 0)$, $B(3, -3)$, $C(1, 1)$, $D(-2, 2)$ lie on a circle.

28. If k is constant, show that

$$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$$

is the equation of a line through the intersection of the lines represented by

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0.$$

29. By using the result in the preceding problem find the equation of the line through the intersection of the lines $2x + y - 5 = 0$, $x - 2y - 5 = 0$ and perpendicular to $y + 3x - 1 = 0$.

30. Show that

$$y = y_1 + \frac{m}{2}[x - x_1 + |x - x_1|]$$

represents a locus which for $x < x_1$ coincides with the horizontal line $y = y_1$ and for $x > x_1$ coincides with

$$y = y_1 + m(x - x_1).$$

31. Determine the equations of the two lines which bisect the angles between

$$x + y - 7 = 0, \quad 7x - y - 1 = 0.$$

32. Determine the lines which bisect the angles between $x - 3y = 0$, $y + 3x = 0$.

33. Find the equations of the tangent and normal to the curve $y = x^3$ at $(2, 8)$.

34. Find the equations of the tangent and normal to the curve $xy = 6$ at $(2, 3)$.

35. Find the equations of the tangent and normal to the curve $y^2 = 4x$ at $(1, -2)$.

36. Find the equations of the tangent and normal to the curve

$$x^2 + xy + y^2 = 3$$

at $(1, 1)$.

37. Determine the equations of the two tangents to the curve

$$y^2 = x^2 - x^3$$

at the origin.

38. Show that the tangent to the curve $y^2 = x^3$ at (x_1, y_1) intersects the curve a second time at $(\frac{1}{4}x_1, -\frac{1}{8}y_1)$.

39. A tangent is drawn to the curve $xy = a$ at (x_1, y_1). Determine the area of the triangle it forms with the coördinate axes.

40. A tangent is drawn to the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

at the point (x_1, y_1). Find the length of the part of this tangent intercepted between the coördinate axes.

41. Find the equations of the two tangents to the curve $y = x^2$ which pass through the point (2, 3).

42. By Newton's method find a root of the equation

$$x^3 + 3x = 15$$

between 2 and 3.

43. By Newton's method find a root of the equation

$$x^5 + 3x + 2 = 0$$

between -1 and 0.

44. By Newton's method find a root of the equation

$$3x^4 - 6x^3 = 1$$

between 2 and 3.

45. Find the angle between the straight line $y - 3x + 2 = 0$ and the curve $y = x^3$ at (1, 1).

46. Find the angle between the curves $y^2 = 4x$ and $x^2 = 4y$ at each intersection.

47. Find the angles at which the straight line $x + 2y = 5$ and the curve $x^2 + y^2 = 5x$ intersect.

48. Show that the curves $xy = a$, $y^2 - x^2 = b$ are orthogonal for all values of the constants a and b.

49. Show that the curves $x^2 + 2y^2 = a^2$, $x^2 = by$ are orthogonal for all values of the constants a and b.

50. Find the perimeter of the triangle with vertices (2, 3), $(-3, 3)$, and (1, 1).

51. Show that (6, 2), $(-2, -4)$, $(5, -5)$, and $(-1, 3)$ are on a circle with center $(2, -1)$.

52. Find the point on the x-axis equidistant from (0, 4) and $(-3, -3)$.

53. Find the equation satisfied by the coördinates x, y of a point equidistant from (0, 2) and $(-3, 4)$.

54. Find the equation of a curve such that any point $P(x, y)$ on it is equidistant from (2, 0) and the y-axis.

55. Find the equation of a curve such that the distance of any point on it from (4, 0) is twice its distance from the origin.

56. Each point on a curve is twice as far from (0, 6) as from the x-axis. Find the equation of the curve.

57. Find the equation of the circle with center (3, 4) and radius 5.

58. Find the equation of the circle having $A(-2, 1)$, $B(4, 3)$ as ends of a diameter.

59. Find the center and radius of the circle

$$x^2 + y^2 + 6x - 2y = 6.$$

60. Find the center and radius of the circle

$$4(x^2 + y^2) - 16x + 12y = 11.$$

61. Show that the equation

$$x^2 + y^2 - 4x + 6y + 13 = 0$$

is satisfied by the coördinates of only one real point.

62. Show that there is no real point which satisfies the equation

$$x^2 + y^2 - 2x + 4y + 8 = 0.$$

63. Find the equation of the circle through $(-2, 4)$ and having the same center as the circle

$$x^2 + y^2 - 6x + 2y + 6 = 0.$$

64. Show that the circles

$$x^2 + y^2 - 2x + 4y + C = 0$$

have the same center for all values of the constant C.

65. Find the equations of the circles through $(1, 2)$ and tangent to both coördinate axes.

66. Find the equations of the circles with centers at the origin and tangent to the circle

$$x^2 + y^2 - 8x + 6y + 9 = 0.$$

67. Given $A(-1, 0)$, $B(1, 0)$, find the equation satisfied by the coördinates x, y of each point $P(x, y)$ on the circular arc APB if the angle from AP to BP is 45°.

68. Find the equation of the circle through the points $(4, 2)$, $(5, -5)$, and $(-3, 1)$.

69. Find the equation of the circle which circumscribes the triangle with vertices $(2, -2)$, $(-4, 6)$, $(3, 5)$.

70. Two circles are represented by the equations

$$x^2 + y^2 + 3x - 2y - 9 = 0,$$

$$x^2 + y^2 - x + 2y - 5 = 0.$$

By subtracting these the equation of a line

$$4x - 4y - 4 = 0$$

is obtained. How is this line related to the two circles?

71. A point $P(x, y)$ moves so that the ratio of its distances from two fixed points is a constant k. If k is different from unity, show that the locus is a circle.

72. Find the locus of points from which tangents to the circles $x^2 + y^2 = 4$, $x^2 + y^2 - 2x + 4y = 4$ are of equal length.

73. If $P_1(x_1, y_1)$ is outside the circle

$$x^2 + y^2 + Ax + By + C = 0,$$

show that

$$x_1^2 + y_1^2 + Ax_1 + By_1 + C$$

is the square of the length of the tangent from P_1 to the circle.

Plot the following parabolas and determine their axes, vertices, and foci:

74. $y^2 = 8x$.
75. $y^2 = -6x$.
76. $y^2 = 4x + 4y$.
77. $x^2 = 6y - 6$.
78. $x^2 - 6x - 4y + 1 = 0$.
79. $x^2 - 12x + 12y = 0$.

80. Find the focus and directrix of the parabola represented by

$$y^2 - 6y + 8x - 7 = 0.$$

81. Find the focus and directrix of the parabola

$$x^2 + 2x - 6y + 13 = 0.$$

82. Show that the parabolas

$$y^2 = 4px + 4p^2$$

have the same focus for all values of p.

83. An arch has the form of a parabola with vertical axis. If the arch is 12 feet high at the center and 30 feet wide at the base, find the height above the base at which the width is 15 feet.

84. An arch in the form of a parabola is 20 feet wide at the bottom, and the highest point is 16 feet above the base. What is the length of the beam placed horizontally across the arch 4 feet from the top?

85. A cable of a suspension bridge hangs in a parabola with vertical axis. The roadway, which is horizontal and 240 feet long, is supported by vertical wires attached to the cable, the longest being 80 feet and the shortest 30 feet. Find the length of the supporting wire 40 feet from the middle.

86. A parabolic arch has base b and altitude h. Determine the area bounded by the arch and base.

87. A parabolic arch of base b and altitude h is rotated about the base. Find the volume generated.

88. The normal to the parabola $y^2 = 4px$ at $P_1(x_1, y_1)$ intersects the x-axis at N. Find the projection of P_1N on the x-axis.

89. Show that the tangent at any point $P(x, y)$ of a parabola makes equal angles with the line parallel to the axis and the line from P to the focus.

Plot the following ellipses and for each determine the center, axes, foci, and eccentricity:

90. $x^2 + 4y^2 = 4$.
91. $4x^2 + 9y^2 = 18y$.
92. $x^2 + 4y^2 + 4x - 8y + 4 = 0$.
93. $16x^2 + 9y^2 = 144$.
94. $3x^2 + 4y^2 - 6x + 4y + 1 = 0$.
95. $9x^2 + 4y^2 - 36x - 36y + 36 = 0$.

96. Find the equation of the ellipse with center $(-2, 4)$ tangent to both coördinate axes.

97. Find the equation of the ellipse with foci $(0, 0)$, $(0, 4)$ and passing through $(3, 0)$.

98. Find the equation of the ellipse with center $(-2, 1)$ and passing through $(0, 4)$ and $(4, 2)$.

99. Find the equation of the line tangent to the ellipse $2x^2 + 3y^2 = 5$ at the point $(1, 1)$.

100. If the ends of a bar move on perpendicular lines, show that a point P of the bar at distances a and b from its ends describes an ellipse with semiaxes a and b.

101. A circle is deformed in such a way that distances of its points from a fixed diameter are all changed in the same ratio. Show that the resulting curve is an ellipse.

102. By using the result in the preceding problem show that the area within the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

103. Show that lines from a point P on an ellipse to its foci make equal angles with the tangent at P.

104. If the sum of the lengths of the tangents from $P(x, y)$ to two fixed circles is a constant, show that the locus of P is an ellipse.

Plot the following hyperbolas and for each determine the center, axes, foci, eccentricity, and asymptotes:

105. $9x^2 - 4y^2 = 36$.

106. $4x^2 - y^2 - 8x + 4y = 4$.

107. $y^2 - 2x^2 + 8x + 2y = 11$.

108. $x^2 - y^2 - 2x - 6y = 18$.

109. Find the equation of the hyperbola with foci (0, 0), (0, 4) if the difference of the distances from the foci to any point on the curve is 2.

110. Find the equations of two hyperbolas with center (2, −1) and semiaxes 1 and 2 parallel to OX and OY respectively.

111. Find the equation of the hyperbola with center (−1, 1), axes parallel to the coördinate axes, and passing through (0, 2) and (1, −4).

112. A point moves so that the product of the slopes of the lines joining it to $(-a, 0)$ and $(a, 0)$ is constant. Show that it describes an ellipse or hyperbola.

113. On a level plane the sound of a rifle and that of the bullet striking the target are heard at the same instant. Find the locus of the hearer.

114. A point $P(x, y)$ moves so that its distance from a fixed point divided by its distance from a fixed straight line is a constant e. Show that the locus is an ellipse, parabola, or hyperbola according as $e < 1$, $e = 1$, or $e > 1$.

115. Two vertices A, B of a triangle are fixed and the third $C(x, y)$ is variable. Find the locus of C if $\angle B = 2\angle A$.

116. Show that the ellipses and hyperbolas obtained by assigning values to the constant k in the equation

$$\frac{x^2}{a^2 - k} + \frac{y^2}{b^2 - k} = 1$$

all have the same foci.

117. If F, F' are the foci and P any point on a hyperbola, show that the tangent at P bisects the angle $F'PF$.

118. What are the coördinates of the points (2, 3), (−4, 5), and (5, −7) referred to parallel axes through the point (3, −2)?

119. Find the equation of the curve

$$x^2 + 4y^2 - 2x + 8y + 1 = 0$$

referred to parallel axes through (1, −1).

120. Find the equation of the curve

$$y^3 - 6y^2 + 3x^2 + 12y - 18x + 35 = 0$$

referred to parallel axes through (3, 2).

121. Transform the equations $x + y - 3 = 0$, $2x + 3y - 4 = 0$ to parallel axes so chosen that the new equations have no constant terms.

122. Transform the equation

$$x^2 + y^2 - 2x + 4y - 6 = 0$$

to parallel axes so chosen that the new equation has no terms of the first degree in x and y.

123. Determine the position of a new origin such that

$$y^2 + 4y - 2x + 3$$

has no constant term or first power of y.

124. Transform the equation

$$x^2 + xy + y^2 = 1$$

to new axes bisecting the angles between the original axes.

125. Determine the equation of the curve

$$\sqrt{3}(y^2 - x^2) - 2xy = 4$$

when referred to axes making an angle of 60° with the original axes.

126. By rotation of the axes change

$$x^2 - y^2 = 4$$

to a form that does not contain x^2 or y^2.

127. Through what angle must the axes be rotated so that

$$x^2 + xy = 1$$

shall be changed to a form that has no xy term?

128. By rotation of the axes reduce

$$x^2 - 2xy + y^2 = 4(x + 7)$$

to a form which has no xy term.

129. Determine the form into which

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$$

is changed when the axes are rotated 45°.

130. Show that

$$dx^2 + dy^2$$

is invariant under a change of rectangular axes.

131. Show that

$$y\,dx - x\,dy$$

is invariant under rotation of the axes about the origin.

Make graphs of the following equations:

132. $y = (x - 1)(x + 2)(x - 3)$.

133. $y = x^2(x + 1)$.

134. $x = y^2(y - 2)$.

135. $y = (x - 2)^3$.

136. $y = x^4 - 16$.

137. $y = x^4 - 5x^2 + 4$.

138. $y + 1 = x(x - 1)^2$.

139. $x - 2 = y^2(y - 1)^2$.

140. $y^2 = x^3$.

141. $y^2 = x^3 - 4x$.

142. $y^2 = 4x - x^3$.

143. $x^2 = y^2 - y^4$.

144. $y^2 = x^4 - x^2$.

145. $y^4 = x^2$.

146. $y = x + \dfrac{1}{x}\cdot$

147. $y = x^2 + \dfrac{1}{x^2}\cdot$

148. $xy = x^3 - 2$.

149. $xy^2 = 1 - x$.

150. $y = \dfrac{1}{x - 1} - \dfrac{1}{x + 1}\cdot$

151. $x^2y^2 + 36 = 4y^2$.

152. $y^2 = x^2\dfrac{a - x}{a + x}\cdot$

153. $y = \dfrac{8a^3}{x^2 + 4a^2}\cdot$

154. $x^3 + y^3 = 1$.

155. $x^4 + y^4 = 1$.

156. $x^{100} + y^{100} = 1$.

157. $x^{\frac{1}{100}} + y^{\frac{1}{100}} = 1$.

158. $y^3 = x^4$.

159. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

160. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$.

CHAPTER V

DETERMINANTS

70. Determinants of the Second Order. If the two lines

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2 \end{aligned} \tag{1}$$

intersect, the coördinates of the point of intersection may be found by solving the two equations simultaneously. To do this eliminate the unknowns one at a time, thus obtaining

$$\begin{aligned} (a_1b_2 - a_2b_1)x &= c_1b_2 - c_2b_1, \\ (a_1b_2 - a_2b_1)y &= a_1c_2 - a_2c_1. \end{aligned} \tag{2}$$

If $a_1b_2 - a_2b_1$ is not zero, these equations give

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \tag{3}$$

as the coördinates of the point of intersection.

Because of the frequency with which such combinations occur, the expression $a_1b_2 - a_2b_1$ is given a special notation,

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \tag{4}$$

and a special name, *determinant*. The numbers, such as a_1, b_1, in a horizontal line constitute a *row*, those, such as b_1, b_2, in a vertical line constitute a *column*. The above determinant, having two rows and two columns, is called a determinant of the *second order*.

In terms of determinants the solution of equations (1) can be written

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{5}$$

The use of determinants makes the solution of two equations in two unknowns, and as we shall see later the solution of n equations in n unknowns, analogous to the solution of one equation

$$ax = c \tag{6}$$

in one unknown. In the latter case the coefficient a of the one unknown may be regarded as a determinant of the first order. The solution

$$x = \frac{c}{a} \tag{7}$$

is a fraction having this determinant as denominator, and with numerator obtained from the denominator by replacing the coefficient of x in the equation by the constant term c. Similarly, each denominator of (5) is the determinant formed from the coefficients of all the unknowns in the equations, and each numerator is formed from its denominator by replacing the corresponding coefficients (a_1 and a_2 in case of x, b_1 and b_2 in case of y) by the constant terms c_1, c_2 respectively.

To show how determinants are used in solving simultaneous linear equations in more than two unknowns it will be necessary first to define determinants of the nth order and obtain some of their properties.

71. General Definition of Determinant. Consider an array of numbers arranged in a square of n rows and n columns. Such an array is called a *square matrix*, and the individual numbers are called its *elements*. For example, if $n = 3$ the matrix has the form

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3. \end{matrix}$$

The rows are considered as arranged in order from the top downward, the columns from left to right.

With such a matrix is associated a determinant of the nth order defined as follows:

(1) Form all possible products of n elements so chosen that no two belong to the same row or column.

(2) Consider the elements in each product as placed in the positions they occupy in the matrix and joined in pairs by straight lines. If the number of those lines which slope upward to the

right is even or zero, precede the product by a + sign; if the number of such lines is odd, precede the product by a − sign.

(3) Each product together with its attached algebraic sign is called a *term*, and the sum of all such terms is called the *determinant* of the matrix.

The determinant is represented by the square array with a vertical bar placed on each side.

As an example take the third-order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The following six products can be formed having as factors one element from each row and one from each column:

$$a_1b_2c_3, \quad b_1c_2a_3, \quad c_1a_2b_3,$$
$$a_1c_2b_3, \quad b_1a_2c_3, \quad c_1b_2a_3.$$

By rule (2) above, the sign to be attached to the first three is +, that to the last three −. Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_1c_2b_3 - b_1a_2c_3 - c_1b_2a_3.$$

72. Properties of Determinants. The following properties of determinants are almost immediate consequences of the definition.

(1) *If all elements of a row or column are zero, the determinant is zero.*

(2) *If all elements of a row or column are multiplied by the same factor k, the determinant is multiplied by k.*

Thus

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(3) *If two rows or two columns are interchanged, the determinant merely changes algebraic sign.*

Thus

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(4) *If two rows or two columns are identical, the determinant is zero.*

Thus
$$\begin{vmatrix} a_1 & b_2 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} = 0,$$
the first and third columns being identical.

(5) *If the corresponding elements of two rows or two columns are proportional, the determinant is zero.*

Thus
$$\begin{vmatrix} 1 & 2 & 3 \\ k & 2k & 3k \\ 7 & 5 & 4 \end{vmatrix} = 0,$$
the second row being k times the first.

(6) *If all elements of a row or column are binomials, the determinant is the sum of two determinants obtained by deleting one term or the other in each binomial.*

Thus
$$\begin{vmatrix} a_1 & x_1 + y_1 & c_1 \\ a_2 & x_2 + y_2 & c_2 \\ a_3 & x_3 + y_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & x_1 & c_1 \\ a_2 & x_2 & c_2 \\ a_3 & x_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & y_1 & c_1 \\ a_2 & y_2 & c_2 \\ a_3 & y_3 & c_3 \end{vmatrix}.$$

(7) *The determinant is not changed in value if to each element of one row is added k times the corresponding element of another row, or to each element of one column is added k times the corresponding element of another column.*

By repeated application of this theorem it may be shown that the determinant does not change value when several rows are multiplied by constants and added to another row, or several columns multiplied by constants and added to another column.

Thus
$$\begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(8) *If the rows are changed into columns without change of order, the value of the determinant is not changed.*

Thus
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

By application of these properties the work of evaluating a determinant can often be greatly reduced.

Example 1. Find the value of

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Adding corresponding elements of the first and second rows, dividing by 4, and subtracting from the third row results in

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Example 2. Find the value of

$$\begin{vmatrix} 3 & 5 & 7 & 1 \\ 4 & 3 & 1 & 2 \\ 2 & 0 & 3 & 1 \\ 6 & -9 & -3 & 3 \end{vmatrix}.$$

Multiply the last column by 2 and subtract from the first, then multiply the second row by 3 and add to the fourth. In this way we obtain

$$\begin{vmatrix} 1 & 5 & 7 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & -9 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 7 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 9 \end{vmatrix}.$$

The only non-vanishing term in the last determinant is

$$1 \cdot 3 \cdot 3 \cdot 9 = 81.$$

73. Minors and Cofactors. In a determinant delete the row and column to which a given element e belongs. The determinant formed from the remaining elements without change of order is called the *minor* of e.

In the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

for example, the minor of a_1 is

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

and that of b_2 is

$$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}.$$

If a determinant contains an element e, certain of its terms contain e as factor. If the sum of such terms is

$$eE,$$

E is called the *cofactor* of e.

The minor and cofactor of a given element e are related in a simple way. To determine this relation suppose first that e is the element in the upper left-hand corner of the determinant. The definition of determinant shows that the minor and cofactor are then equal. Suppose next that e is the element in the rth row and sth column. By $r + s - 2$ interchanges of two adjacent rows or two adjacent columns it can be moved to the upper left-hand position. These moves leave the minor unchanged but change the sign of the determinant and so that of the cofactor $r + s - 2$ times. For the element in the rth row and sth column we thus have

$$\text{cofactor} = (-1)^{r+s-2}\,(\text{minor}) = (-1)^{r+s}\,(\text{minor})$$

In the third-order determinant above, for example, the cofactor of a_1 is

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$

and that of b_1 is

$$-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

In the applications of determinants the following properties of cofactors are fundamental:

(1) *The sum of the products of the elements in any row, or column, by their cofactors is equal to the determinant.*

For, if $a_1, a_2, \cdots, a_n$ are the elements of a row or column, and $A_1, A_2, \cdots, A_n$ are their cofactors, then a_1A_1 is the sum of terms containing a_1, a_2A_2 the sum of terms containing a_2, etc. Since every term contains an element from the given row, or column, the sum of all terms is

$$a_1A_1 + a_2A_2 + \cdots + a_nA_n.$$

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

(2) *The sum of the products of elements in any row (column) by the cofactors of corresponding elements in another row (column) is zero.*

Thus, if $a_1, a_2, \cdots, a_n$ are elements in one column and $B_1, B_2, \cdots, B_n$ cofactors of corresponding elements in another column

$$a_1B_1 + a_2B_2 + \cdots + a_nB_n = 0.$$

For, the sum on the left may be obtained from the expansion of the determinant

$$b_1B_1 + b_2B_2 + \cdots + b_nB_n$$

by replacing $b_1, b_2, \cdots, b_n$ by $a_1, a_2, \cdots, a_n$, and so is a determinant with two columns equal to $a_1, a_2, \cdots, a_n$ (§72, 4).

74. Solution of n Linear Equations in n Unknowns. As an illustration take three equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3, \end{aligned} \tag{1}$$

in three unknowns x_1, x_2, x_3. Let A_{rs} be the cofactor of a_{rs} in the determinant

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{2}$$

formed from the coefficients of all the unknowns. Multiply the first equation by A_{11}, the second by A_{21}, the third by A_{31}, and add. By theorems 1 and 2, §73, the coefficient of x_1 in the sum is

$$a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = A, \tag{3}$$

and the coefficients of x_2 and x_3 are zero. Also the right side

$$c_1A_{11} + c_2A_{21} + c_3A_{31} \tag{4}$$

is obtained from A by replacing a_{11}, a_{21}, a_{31} by c_1, c_2, c_3, respectively. The sum is therefore

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} x_1 = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}.$$

If A is not zero, we thus have

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \tag{5}$$

and similar expressions are obtained for x_2 and x_3.

That these values actually satisfy equations (1) may be shown by direct substitution. Thus if we substitute in

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - c_1 = 0$$

and clear fractions the left side will be found to be a determinant of the fourth order with two rows equal to a_{11}, a_{12}, a_{13}, c_1, and therefore zero (§72, 4).

A similar argument is applicable to n equations in n unknowns, thus proving the following, known as *Cramer's rule*:

A set of n equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2, \\ \cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n, \end{aligned}$$

in n unknowns x_1, x_2, $\cdots$, x_n has one and only one solution if the determinant

$$A = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \cdots\cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}$$

is not zero. Each unknown is a fraction with denominator A and numerator obtained from A by replacing the coefficients of that unknown by the corresponding constant terms c_1, c_2, $\cdots$, c_n.

Example. Show that the equations

$$\begin{aligned} x + y + z &= 2, \\ 2x - y - z &= 1, \\ x + 3y + 4z &= 3, \end{aligned}$$

have a unique solution, and find the value of y.

The determinant of the coefficients is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{vmatrix} = -3.$$

Since this is not zero, the equations have a unique solution. By Cramer's rule, the value of y is

$$y = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{vmatrix}} = \frac{-6}{-3} = 2.$$

75. Homogeneous Linear Equations. Linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0, \end{aligned} \tag{1}$$

with constant terms all zero are called *homogeneous*. The values

$$x_1 = x_2 = \cdots = x_n = 0 \tag{2}$$

constitute an obvious but usually trivial solution. The question of importance is whether there is any other solution.

For a system of n homogeneous equations in n unknowns to have a solution in which the unknowns are not all zero it is necessary and sufficient that the determinant of the coefficients be zero.

To prove this we reduce the equations to simpler form by the process of elimination ordinarily used in solving such equations. For this purpose, take any one of the equations, which we call equation (1), and designate by x_1 any unknown in it with coefficient not zero. By multiplying this equation by appropriate constants and subtracting eliminate x_1 from all the other equations. Next, take any one of the remaining equations, which we call equation (2), designate by x_2 one of its unknowns with coefficient not zero, and by means of this equation eliminate x_2 from all the other equations, including number (1). Continue this process until the remaining equations, if any, do not have any coefficients different from zero. Suppose that this happens after r equations have been treated in the above way. The final equations have the

same solutions as the original ones, for in each change we can re-obtain the preceding equations by merely adding proper multiples of one equation to the others. Also these operations do not change the determinant of the coefficients (except possibly in sign, owing to change of order), for in each case the change consists in subtracting a multiple of one row from another row (§72, 7). The above process can end in one or other of two ways. First, if $r = n$, the final equations take the form

$$b_1x_1 = 0, \quad b_2x_2 = 0, \ \cdots, b_nx_n = 0$$

with b_1, b_n, $\cdots$, b_n all different from zero. The only solution is then

$$x_1 = x_2 = \cdots = x_n = 0,$$

and the determinant

$$b_1b_2b_3\cdots b_n$$

is not zero. Second, if $r < n$, arbitrary values can be assigned to x_{r+1}, $\cdots$, x_n, and x_1, x_2, $\cdots$, x_r can then be determined from the first r equations. In this case the determinant is zero since all elements in the $r + 1$st row are zero. Thus there are solutions different from zero when the determinant is zero and none when the determinant is not zero, which was to be shown.

Example 1. By means of a determinant express the condition that three points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a line.

If

$$Ax + By + C = 0$$

is the equation of a line through the three points,

$$\begin{aligned} Ax_1 + By_1 + C &= 0, \\ Ax_2 + By_2 + C &= 0, \\ Ax_3 + By_3 + C &= 0. \end{aligned}$$

These equations are homogeneous in the unknowns A, B, C. For a solution to exist with A, B, C not all zero, the coefficient determinant must be zero. Thus

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \tag{3}$$

is the condition required.

Example 2. Find the equation of the circle through three points (x_1, y_1), (x_2, y_2), (x_3, y_3) not on a line.

Let

$$A(x^2 + y^2) + Bx + Cy + D = 0 \tag{4}$$

be the equation required. Since the circle passes through the three points

$$\begin{aligned} A(x_1^2 + y_1^2) + Bx_1 + Cy_1 + D &= 0,\\ A(x_2^2 + y_2^2) + Bx_2 + Cy_2 + D &= 0,\\ A(x_3^2 + y_3^2) + Bx_3 + Cy_3 + D &= 0. \end{aligned}$$

These equations and (4), being homogeneous in four unknowns A, B, C, D, must have a coefficient determinant equal to zero. Thus

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{5}$$

When expanded this has the form (4) and so represents a circle through the three points provided that

$$A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

is not zero, that is, provided that (x_1, y_1), (x_2, y_2), (x_3, y_3) are not on a line. If, however, the points are on a line, $A = 0$ and (5) is the equation of the line.

PROBLEMS

Solve by determinants:

1. $\begin{cases} 3x + y = 1, \\ x - 2y = 5. \end{cases}$

2. $\begin{cases} x + 2y = 4, \\ 2x - y = 3. \end{cases}$

3. $\begin{cases} x + 3y - 1 = 0, \\ 3x - y + 7 = 0. \end{cases}$

4. $\begin{cases} 3x - 2y - 2 = 0, \\ 5x + 4y + 4 = 0. \end{cases}$

Evaluate the following determinants:

5. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 2 & -1 & 4 \\ 1 & 3 & 2 \\ 5 & 1 & 10 \end{vmatrix}$

7. $\begin{vmatrix} 2 & 4 & -5 \\ -3 & -1 & 7 \\ 4 & 8 & -10 \end{vmatrix}$

8. $\begin{vmatrix} 3 & -1 & 2 \\ 1 & 2 & -3 \\ -2 & 3 & 1 \end{vmatrix}$

9. $\begin{vmatrix} 1 & 2 & -1 & 3 \\ -1 & 4 & 2 & 1 \\ 3 & 1 & -3 & 2 \\ 2 & 3 & 1 & 4 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 2 & 3 & -1 \\ -1 & 4 & 5 & 1 \\ 2 & 6 & 7 & -2 \\ -3 & 8 & 9 & 3 \end{vmatrix}$

Solve the following systems of equations:

11. $\begin{cases} x + 2y - z = 1, \\ 2x - y + 3z = 2, \\ 3x - 2y + z = 4. \end{cases}$

12. $\begin{cases} x - 2y - z = 2, \\ 2x - 3y - 2z = 1, \\ 3x - 2y - 2z = 4. \end{cases}$

13. $\begin{cases} x - y + z = 4, \\ 2x + 3y + 2z = 3, \\ 4x + 2y + 3z = 8. \end{cases}$

14. $\begin{cases} x + y + z - 9 = 0, \\ x - y - z + 3 = 0, \\ x + y - z - 5 = 0. \end{cases}$

15. $\begin{cases} x + y + z + u = 5, \\ x - 2y + 3z + u = 1, \\ 3x + y - z + 2u = 4, \\ 2x + 3y + 2z - u = 3. \end{cases}$

16. $\begin{cases} x + y + z - u = 1, \\ 2x - y - z + 3u = 8, \\ 3x + 2y + z - u = 6, \\ x - 3y - z + 4u = 7. \end{cases}$

17. Show that the equations

$$x + y = 1, \quad x + y = 2$$

have no simultaneous solutions.

18. If

$$a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0,$$

show that

$$x : y : z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

19. Given

$$\begin{aligned} x + y - 2z &= 0, \\ 5x + y + z &= 0, \end{aligned}$$

find three numbers proportional to x, y, z.

Show that each of the following sets of equations has solutions in which the unknowns are not all zero, and find three numbers proportional to x, y, z:

20. $\begin{cases} x + y + z = 0, \\ x + 4y + 3z = 0, \\ 5x - y + z = 0. \end{cases}$

21. $\begin{cases} 2x - y + 3z = 0, \\ 4x + 3y + 6z = 0, \\ 6x - 5y + 9z = 0. \end{cases}$

22. $\begin{cases} x + y - z = 0, \\ x - 2y - 4z = 0, \\ y + z = 0. \end{cases}$

23. $\begin{cases} x + 3y = 0, \\ 2y + z = 0, \\ 2x - 3z = 0. \end{cases}$

24. Show that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is the equation of the straight line through (x_1, y_1) and (x_2, y_2).

25. Show that

$$\begin{vmatrix} y^2 & x & 1 \\ 4 & 1 & 1 \\ 9 & -1 & 1 \end{vmatrix} = 0$$

is the equation of the parabola with axis parallel to the x-axis and passing through $(1, 2)$, $(-1, 3)$, $(-1, -3)$.

26. Find the equation of the parabola with axis parallel to the x-axis and passing through $(2, 3)$, $(-3, 2)$, $(-3, -2)$.

27. Find the equation of the circle which passes through (0, 1), (4, 0), and (2, 5).

28. Determine whether the four points (4, 2), (5, 1), (−2, 6), and (−3, −5) are on a circle.

29. If the three lines

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ a_3x + b_3y + c_3 &= 0 \end{aligned}$$

pass through a point, show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Is it true conversely that if the determinant is zero the lines pass through a point? Consider the case of parallel lines.

30. Determine whether the lines

$$\begin{aligned} x - 2y + 1 &= 0, \\ 2x + y - 1 &= 0, \\ 3x + 4y - 3 &= 0, \end{aligned}$$

pass through a point.

31. If the equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ a_3x + b_3y + c_3 &= 0, \end{aligned}$$

have a simultaneous solution, show that constants k_1, k_2, k_3, not all zero, exist such that

$$k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) + k_3(a_3x + b_3y + c_3) = 0$$

for all values of x and y. Is the converse true? Take, for example, the equations $x + y - 1 = 0$, $x + y - 2 = 0$, $x + y - 3 = 0$.

32. Determine numbers k_1, k_2, k_3 such that

$$k_1(2x + 3y - z) + k_2(3x + 4y - 3z) + k_3(x + 3y + 4z) = 0$$

for all values of x, y, z.

CHAPTER VI

TRIGONOMETRIC FUNCTIONS

76. Sine Curves. If a, b, c are constants the graph of the equation

$$y = a \sin (bx + c) \tag{1}$$

is called a *sine curve.*

Unless the contrary is explicitly stated it is assumed that the angle $bx + c$ is measured in radians. When x changes, this angle changes. As the angle varies from 0 to 2π the sine runs through its entire range of values and the point (x, y) traces a complete *wave* AC of the curve (Figure 103). As the angle continues to

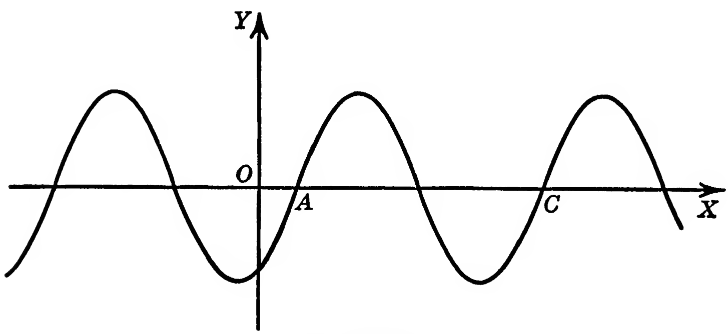

FIGURE 103.

increase, the sine runs repeatedly through the same range of values. Thus the entire curve consists of an infinite number of such waves.

The distance between points at which the angles differ by 2π is called the *wave length.* In a displacement between two such points

$$\Delta(bx + c) = b\Delta x = 2\pi,$$

whence

$$\text{wave length} = |\Delta x| = \frac{2\pi}{|b|}. \tag{2}$$

The sine has its maximum value 1 when the angle is $\frac{\pi}{2}$, and its

minimum -1 when the angle is $-\frac{\pi}{2}$. The curve thus lies between the lines $y = \pm a$, touching one of these lines at the middle of each half wave. The coefficient a measures the height of the wave crest above the axis $y = 0$.

The curve crosses the x-axis at the point where

$$bx + c = 0.$$

A change in c merely changes the position

$$x = -\frac{c}{b}$$

at which this crossing occurs.

Since

$$\cos(bx + c) = \sin\left(bx + c + \frac{\pi}{2}\right),$$

the equation

$$y = a\cos(bx + c)$$

also represents a sine curve, namely the curve obtained by moving (1) a quarter wave length parallel to the axis.

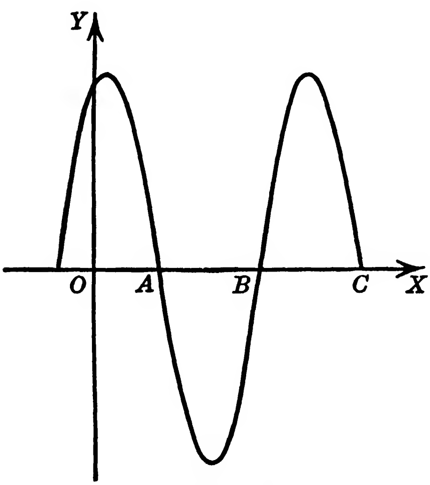

FIGURE 104.

Example. Plot the curve

$$y = 2\sin(2 - 3x).$$

This curve lies between the lines $y = \pm 2$. It crosses the x-axis at the point A (Figure 104) where $2 - 3x = 0$,

$$x = \tfrac{2}{3}.$$

When x increases from this value the sine becomes negative and so the curve slopes downward. A complete wave length is

$$AC = \frac{2\pi}{3}.$$

At the point B, halfway between A and C, the angle is $-\pi$ and y is zero. The half waves AB, BC can now be sketched and the remainder of the curve obtained by indefinite repetition.

77. Periodic Functions. If k is constant and

$$f(x + k) = f(x) \tag{1}$$

for all values of x, the function $f(x)$ is said to be *periodic* and the number k is called its *period*. If the part of such a curve between $x = x_1$ and $x = x_1 + k$ is constructed, the entire curve is the infinite set of parts obtained by moving this one to the left or to the right through the distances k, $2k$, etc.

Thus $\sin (bx + c)$ is a periodic function of period $\dfrac{2\pi}{b}$. The cosine, tangent, cotangent, secant, and cosecant of $bx + c$ are also periodic functions of x with the same period. In addition $\tan (bx + c)$ and $\cot (bx + c)$ repeat their values when the angle is increased by π. These functions therefore have the shorter period $\dfrac{\pi}{b}$.

A function of period k has also the periods $2k$, $3k$, etc. Sometimes an expression consists of parts each of which is periodic but with different periods. If these periods are commensurable, their least common multiple is a period for the entire expression.

In the function

$$f(x) = \sin \frac{x}{2} + \tan \frac{x}{3},$$

for example, the sine is of period 4π and the tangent of period 3π. The sum is periodic with period equal to the least common multiple of these, namely, 12π.

Example 1. $y = \tan 2x$.

When the angle $2x$ is increased by π $\left(x \text{ increased by } \dfrac{\pi}{2}\right)$ the tangent of $2x$ is not changed. The function is thus periodic of period $\frac{1}{2}\pi$. The curve therefore consists of a series of parts each of which is obtained by moving the preceding one a distance $\frac{1}{2}\pi$ to the right. One of these branches passes through the origin. As x increases from 0 to $2x = \dfrac{\pi}{2}$, y increases from 0 to infinity. The line $x = \frac{1}{4}\pi$ is therefore an asymptote. As x decreases to $-\frac{1}{4}\pi$, y decreases to $-\infty$. The branch through the origin thus extends from $x = -\frac{1}{4}\pi$, $y = -\infty$, to $x = \frac{1}{4}\pi$, $y = \infty$. The whole curve is a series of such branches at distances $\frac{1}{2}\pi$ apart (Figure 105).

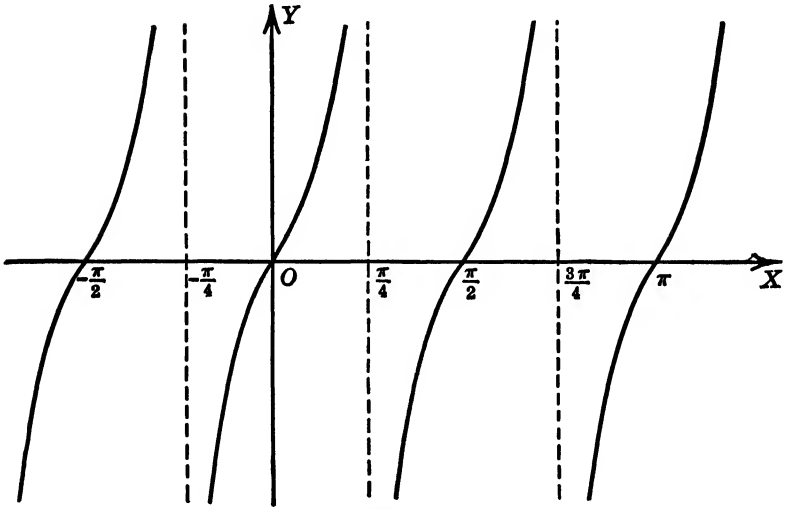

FIGURE 105.

Example 2. $y = \sec x$. The secant of an angle is in absolute value never less than 1. Consequently the curve lies outside the lines $y = \pm 1$. Since the period is 2π it will be sufficient to plot the portion between

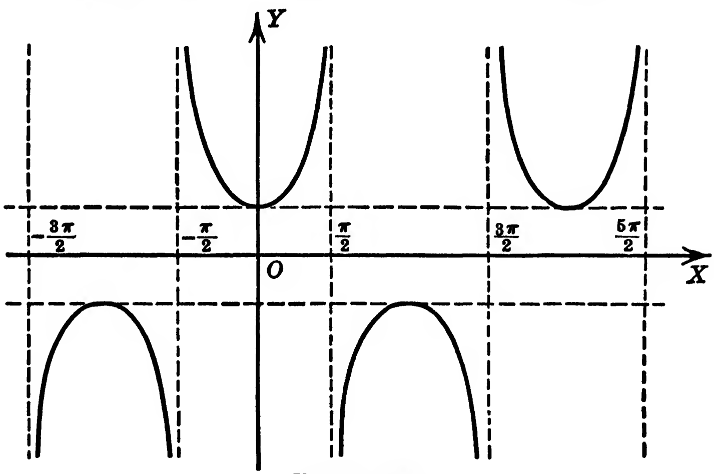

FIGURE 106.

$x = -\dfrac{\pi}{2}$ and $x = \dfrac{3}{2}\pi$. The most important points on this part of the curve are given in the following table:

$x =$	$-\dfrac{\pi}{2}$,	0,	$\dfrac{\pi}{2}$,	π,	$\dfrac{3}{2}\pi$,
$y =$	$-\infty, \infty$,	1,	$\infty\ -\infty$,	-1,	$-\infty, \infty$,

the expression $-\infty, \infty$ being used to signify that on the left of this position y becomes indefinitely large and negative, and on the right indefinitely large and positive. The curve is a series of U-shaped branches alternately above $y = 1$ and below $y = -1$ (Figure 106).

Example 3. $y = 2 \sin \frac{1}{2}x + 6 \cos \frac{1}{4}x$. To construct this curve draw the curves

$$y_1 = 2 \sin \tfrac{1}{2}x, \quad y_2 = 6 \cos \tfrac{1}{4}x,$$

and on each vertical line determine the point $P(x, y)$ such that $y = y_1 + y_2$, that is (Figure 107) such that

$$MP = MP_1 + MP_2.$$

To leave $\sin \frac{1}{2}x$ and $\cos \frac{1}{4}x$ both unchanged x must be increased by 8π

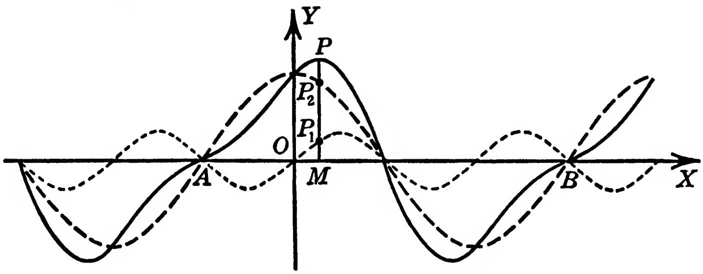

FIGURE 107.

or a multiple of 8π. Hence it is sufficient to plot the section AB from $x = -2\pi$ to $x = 6\pi$. The whole curve is a series of such sections placed end to end.

78. Differentiation of Trigonometric Functions. Let u be the independent variable or any differentiable function of the variable.

VII. $d \sin u = \cos u \, du.$

VIII. $d \cos u = -\sin u \, du.$

IX. $d \tan u = \sec^2 u \, du.$

X. $d \cot u = -\csc^2 u \, du.$

XI. $d \sec u = \sec u \tan u \, du.$

XII. $d \csc u = -\csc u \cot u \, du.$

The negative sign occurs in the differentials of all cofunctions. This is to be expected since for angles in the first quadrant the cofunctions diminish as the angle increases.

Formulas VII and VIII were proved in §22. The other formulas may be proved by expressing the functions in terms of sine and cosine.

Thus to prove IX we write

$$\tan u = \frac{\sin u}{\cos u}.$$

Differentiating this as a fraction and using VII and VIII, we have

$$\begin{aligned} d \tan u &= \frac{\cos u \, d(\sin u) - \sin u \, d(\cos u)}{\cos^2 u} \\ &= \frac{\cos^2 u \, du + \sin^2 u \, du}{\cos^2 u} \\ &= \sec^2 u \, du. \end{aligned}$$

Example 1. $y = \sec 2x + \tan 2x.$
By IX and XI,

$$\begin{aligned} dy &= \sec 2x \tan 2x \, d(2x) + \sec^2 2x \, d(2x) \\ &= 2 \sec 2x(\tan 2x + \sec 2x) \, dx. \end{aligned}$$

Example 2. $y = \sin^4 (3x - 2).$
This is equivalent to

$$y = [\sin (3x - 2)]^4 = u^4.$$

The derivative is therefore

$$\begin{aligned} \frac{dy}{dx} &= 4u^3 \frac{du}{dx} \\ &= 4 \sin^3 (3x - 2) \frac{d}{dx} \sin (3x - 2) \\ &= 12 \sin^3 (3x - 2) \cos (3x - 2). \end{aligned}$$

Example 3. $y = x + \cot x - \frac{1}{3} \cot^3 x.$
By formula X the derivative is

$$\begin{aligned} \frac{dy}{dx} &= 1 - \csc^2 x + \cot^2 x \csc^2 x \\ &= - \cot^2 x + \cot^2 x \csc^2 x \\ &= \cot^2 x(\csc^2 x - 1) \\ &= \cot^4 x. \end{aligned}$$

79. Inverse Trigonometric Functions. For values of x numerically less than 1 the equation

$$\sin y = x$$

determines y as a function of x. This function is represented by the notation

$$y = \sin^{-1} x$$

and is called the angle whose sine is x, or the *inverse sine* of x. Since the angle is equal to an arc on a unit circle the equivalent notation

$$\sin^{-1} x = \text{arc} \sin x$$

is also often used.

The relation of x and y is shown graphically in Figure 108. For a given value of x (numerically less than 1) there are infinitely many values of y, represented by the points at which the vertical line of abscissa x intersects the curve. We express this by saying the inverse sine of x is a multivalued function of x, in fact, an infinitely multivalued function. To avoid ambiguity the notation $\sin^{-1} x$ is therefore usually restricted to represent a particular one of these values, namely the one between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This is called the *principal value*. In Figure 108 it is represented by the part AB of the curve.

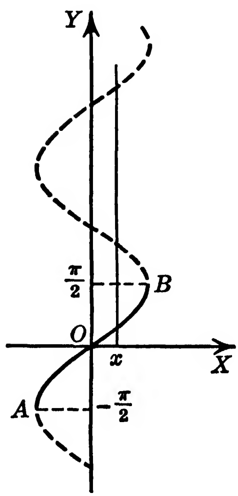

FIGURE 108.

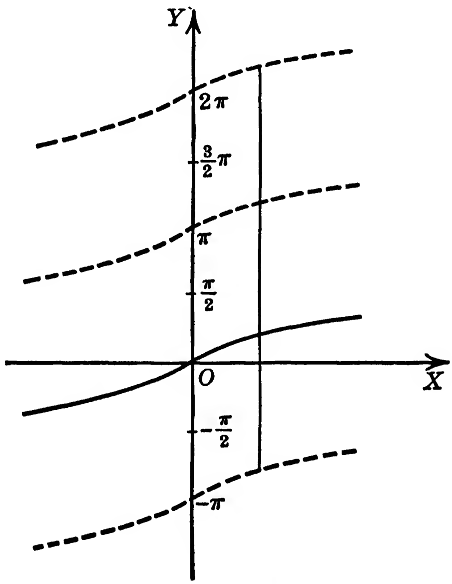

FIGURE 109.

Similar definitions apply to $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$. Each of these notations usually represents a principal value, selected partly for simplicity and partly to make the formulas of differentiation take definite form.

For $\sin^{-1} x$ and $\tan^{-1} x$ the principal value is taken as the angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In Figures 108 and 109 these values are represented by the solid lines.

For $\cos^{-1} x$ and $\cot^{-1} x$ the principal value is the angle in the interval $(0, \pi)$.

For $\sec^{-1} x$ and $\csc^{-1} x$ the principal value is taken in the interval $\left(0, \frac{\pi}{2}\right)$ if $x > 0$ and in the interval $\left(-\pi, -\frac{\pi}{2}\right)$ if $x < 0$.

80. Formulas for Differentiating Inverse Trigonometric Functions. The following formulas give the differentials of the principal values. If other values of the inverse trigonometric functions are used, the algebraic signs given in the formulas may not be correct.

$$\text{XIII.}\quad d \sin^{-1} u = \frac{du}{\sqrt{1 - u^2}}.$$

$$\text{XIV.}\quad d \cos^{-1} u = -\frac{du}{\sqrt{1 - u^2}}.$$

$$\text{XV.}\quad d \tan^{-1} u = \frac{du}{1 + u^2}.$$

$$\text{XVI.}\quad d \cot^{-1} u = -\frac{du}{1 + u^2}.$$

$$\text{XVII.}\quad d \sec^{-1} u = \frac{du}{u\sqrt{u^2 - 1}}.$$

$$\text{XVIII.}\quad d \csc^{-1} u = -\frac{du}{u\sqrt{u^2 - 1}}.$$

It should be noted that the differential of each cofunction merely differs in algebraic sign from that of the corresponding function. This follows since function and cofunction are complementary angles. Thus

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x.$$

To prove any one of the formulas we change from inverse to direct functions. Thus to prove XIII let

$$y = \sin^{-1} u.$$

Then

$$\sin y = u.$$

Differentiation gives

$$\cos y\, dy = du.$$

If y is the principal value $\cos y$ is positive and

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - u^2}.$$

Thus

$$dy = \frac{du}{\cos y} = \frac{du}{\sqrt{1 - u^2}},$$

which was to be proved.

Example 1. $y = \sin^{-1} \dfrac{x - 1}{x + 1}$.

By formula XIII

$$dy = \frac{d\left(\dfrac{x-1}{x+1}\right)}{\sqrt{1 - \left(\dfrac{x-1}{x+1}\right)^2}} = \frac{2\, dx}{(x+1)^2\sqrt{1 - \left(\dfrac{x-1}{x+1}\right)^2}} = \frac{dx}{(x+1)\sqrt{x}}.$$

Example 2. $y = \sqrt{x^2 - a^2} - a \sec^{-1} \dfrac{x}{a}$.

Considering a as constant, we have

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x^2 - a^2} - \frac{a \dfrac{d}{dx}\left(\dfrac{x}{a}\right)}{\dfrac{x}{a}\sqrt{\left(\dfrac{x}{a}\right)^2 - 1}}$$

$$= \frac{x}{\sqrt{x^2 - a^2}} - \frac{a^2}{x\sqrt{x^2 - a^2}}$$

$$= \frac{\sqrt{x^2 - a^2}}{x}.$$

81. Angular Velocity and Acceleration. Consider a body rotating about a fixed axis OA (Figure 110). Let OM, OP be lines perpendicular to the axis, OM being fixed in space and OP rotating with the body, and let θ be the angle from OM to OP. This angle is considered positive when described

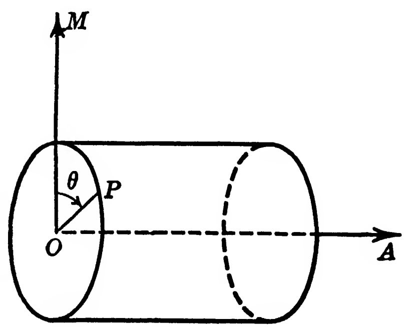

FIGURE 110.

in one direction about OA, negative when in the other. The positive direction is often indicated by an arrow along the axis pointing in the direction a right-hand screw advances when given the positive rotation.

The angle θ is a function of the time. Its derivative

$$\omega = \frac{d\theta}{dt}$$

is called the *angular velocity*, and its second derivative

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

the *angular acceleration*, of the body.

Rotation about a fixed axis is analogous to translation along a fixed straight line; angle, angular velocity, and angular acceleration corresponding to displacement, linear velocity, and linear acceleration.

Example. A body set rotating by a constant torque about a fixed axis has constant angular acceleration α about the axis. If it starts from rest, find the angle turned through in t seconds.

In this case

$$\frac{d^2\theta}{dt^2} = \alpha = \text{constant.}$$

Integrating and determining the constants so that θ and ω are zero when $t = 0$, we obtain

$$\omega = \frac{d\theta}{dt} = \alpha t,$$

$$\theta = \tfrac{1}{2}\alpha t^2.$$

82. Simple Harmonic Motion. The motion of a particle along a line is called *simple harmonic* if the displacement of the particle from a fixed point of the line is represented by an expression of the form

$$s = A \sin (bt + c), \tag{1}$$

where A, b, c, are constants and t is the time.

As the time increases, the angle $bt + c$ varies steadily, and s varies periodically between $-A$ and A. The motion is thus an oscillation or vibration back and forth along the line, A being the

maximum displacement from the center $s = 0$. The numerical value of A is called the amplitude of vibration.

The angle $bt + c$ is called the *phase*, or *phase angle*. As the phase increases from 0 to 2π the displacement s runs through its complete range of values. Such a portion of the motion is called a *complete vibration*. The time required for one vibration

$$T = \frac{2\pi}{b} \tag{2}$$

is the *period*, and the number of vibrations per unit time

$$n = \frac{1}{T} = \frac{b}{2\pi} \tag{3}$$

is the *frequency* of vibration.

The velocity of a particle in simple harmonic motion is

$$v = \frac{dx}{dt} = Ab \cos (bt + c), \tag{4}$$

and its acceleration is

$$a = -Ab^2 \sin (bt + c). \tag{5}$$

Comparison with (1) shows that the acceleration can be written

$$a = -b^2 s. \tag{6}$$

The acceleration is thus proportional to the displacement from the center of motion and is oppositely directed.

A force F acting on a particle of mass m gives it an acceleration determined by the equation

$$F = ma.$$

If such a particle is describing the simple harmonic motion (1) it is then subject to a force

$$F = ma = -mb^2 s$$

proportional to the displacement s and directed toward the center.

Example 1. A particle P moves with constant speed around a circle of radius r making n complete revolutions per second. Determine the motion of its projection on a fixed diameter of the circle.

Let OX (Figure 111) be the fixed diameter. If

$$\theta = XOP$$

is zero at time t_0, its value at any other time t is

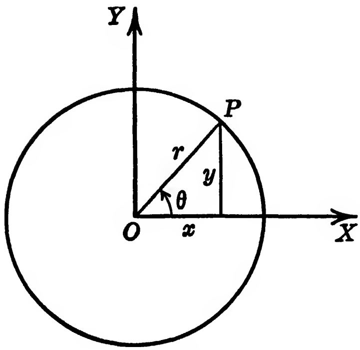

FIGURE 111.

$$\theta = 2n\pi(t - t_0).$$

The projection of P on the x-axis is

$$x = r\cos\theta = r\cos 2n\pi(t - t_0)$$

$$= r\sin\left(2n\pi(t - t_0) + \frac{\pi}{2}\right).$$

The projection thus has simple harmonic motion of frequency n and amplitude r. The projection on the y-axis is

$$y = r\sin 2n\pi(t - t_0).$$

This also has simple harmonic motion of the same amplitude and frequency but with phase at each instant $\frac{\pi}{2}$ less than the projection on the x-axis.

Example 2. The displacement of a particle at time t is

$$s = A\sin bt + B\cos bt.$$

Show that its motion is simple harmonic.

Construct a right triangle of sides A, B, and let θ be the angle opposite B. The hypotenuse of the triangle is

$$C = \sqrt{A^2 + B^2},$$

and

$$A = C\cos\theta, \quad B = C\sin\theta.$$

Consequently

$$s = C(\sin bt\cos\theta + \cos bt\sin\theta)$$

$$= C\sin(bt + \theta).$$

The motion is therefore simple harmonic with amplitude

$$C = \sqrt{A^2 + B^2}$$

and period

$$T = \frac{2\pi}{b}.$$

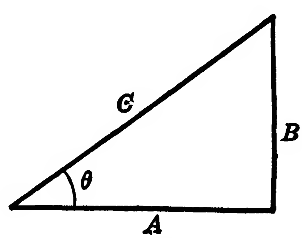

FIGURE 112.

PROBLEMS

Plot the graphs of the following equations:

1. $y = 2 \sin \frac{x}{3}$.
2. $y = 3 \sin 2x$.
3. $y = 4 \cos \frac{1}{2}x$.
4. $y = 2 \cos \left(x - \frac{\pi}{4}\right)$.
5. $y = 3 \sin (x - 1)$.
6. $y = \sin x + \sin 2x$.
7. $y = \sin^2 x$.
8. $y = \tan x$.
9. $y = \csc x$.
10. $y = \cot x$.
11. $y = \tan^2 x$.
12. $y = x \sin x$.
13. $y = x \sin \frac{1}{x}$.
14. $x^2 = \sin y$.

Find $\frac{dy}{dx}$ in each of the following:

15. $y = 4 \sin 3x + 3 \cos 4x$.
16. $y = 3 \sin^2 \frac{x}{2}$.
17. $y = 2 \cos^2 3x$.
18. $y = \frac{1}{2}x - \frac{1}{4} \sin 2x$.
19. $y = x - \sin x \cos x$.
20. $y = 2 \cos \frac{1}{2}x + x \sin \frac{1}{2}x$,
21. $y = \sin 3x - \sin^3 x$.
22. $y = \cos^3 \frac{x}{3} - 3 \cos \frac{x}{3}$.
23. $y = \sin 5x - \frac{1}{3} \sin^3 5x$.
24. $y = x(\frac{1}{3} \cos^3 x - \cos x) + \frac{1}{9} \sin^3 x + \frac{2}{3} \sin x$.
25. $y = 2 \tan 3x$.
26. $y = 3 \cot (1 - 4x)$.
27. $y = \sqrt{\tan x}$.
28. $y = \tan x - x$.
29. $y = 3x + \cot 3x$.
30. $y = \frac{1}{4} \tan^2 2x$.
31. $y = \sec^2 3x - \tan^2 3x$.
32. $y = \sec^4 x - \tan^4 x$.
33. $y = (\sec x + \tan x)^3$.
34. $y = \tan 2x + \frac{1}{3} \tan^3 2x$.
35. $y = \frac{3}{5} \tan^5 \frac{x}{3} + \tan^3 \frac{x}{3}$.
36. $y = \sec^3 \frac{2}{3}x - 3 \sec \frac{2}{3}x$.
37. $y = \frac{1}{5} \sec^5 x - \frac{2}{3} \sec^3 x + \sec x$.
38. $y = \frac{1}{5} \cot^5 x - \frac{1}{3} \cot^3 x + \cot x + x$.
39. $xy + \tan (xy) = 0$.
40. If $y = A \cos nx + B \sin nx$, where A, B, n are constants, show that

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

41. Find the angle at which the curve

$$y = \frac{1}{2} \sin 2x$$

crosses the x-axis.

42. Find the angle between the x-axis and the curve $y = \cot x$ at each intersection.

43. Find the angles at which the curves $y = \tan x$, $y = 2 \sin x$ intersect.

44. Find the maximum height of the curve

$$y = 3 \cos x - 4 \sin x$$

above the x-axis.

45. Find the maximum value of the function $2 \sin x + \cos 2x$.

46. By Newton's method find an approximate solution of the equation

$$\tan x = \tfrac{3}{2}x$$

near $x = \dfrac{\pi}{3}$.

47. Find the area bounded by the x-axis and one arch of the sine curve

$$y = A \sin (bx + c).$$

48. A revolving light 5 miles from a straight shoreline makes 6 revolutions per minute. Find the velocity along the shore of the beam of light at the instant when it makes an angle of 60° with the shoreline.

49. An airplane 10,000 feet above the earth is flying horizontally directly away from an observer. If the angle of elevation is 60° and this angle is diminishing 0.03 radian per second, determine the speed of the plane and the rate at which its distance from the observer is increasing.

50. A right circular cone of vertical angle 2θ is inscribed in a sphere of radius a. For what value of θ is the lateral surface area of the cone greatest?

51. The lower corner of a page is folded over so as just to reach the inner edge. If the width of the page is 6 inches, find the minimum length of crease.

52. A rope with a ring at one end is looped over two pegs in a horizontal line and held taut by a weight fastened to the free end. If the rope slips freely, the weight will descend as far as possible. Find the angle formed at the bottom of the loop.

53. A gutter is made by bending into shape a strip of copper so that the cross section is an arc of a circle. If the width of the strip is a, find the radius of the circle when the carrying capacity is greatest.

54. A vertical telephone pole at a bend in the line is supported from tipping over by a guy wire 40 feet long attached to a stake in the ground. At what angle should the guy wire be inclined to make the tension in it least? Observe that the moment (§134) of the tension about the foot of the pole is constant.

Plot the graphs of the following equations, indicating the principal value by a solid line and dotting the remainder of the curve:

55. $y = \cos^{-1} \tfrac{1}{2}x$.

56. $y = 2 \tan^{-1} 3x$.

57. $y = \cot^{-1} x$.

58. $y = \sec^{-1} x$.

Find $\dfrac{dy}{dx}$ in each of the following problems:

59. $y = 3 \sin^{-1} 4x$.

60. $y = 4 \cos^{-1} \dfrac{x}{2}$.

61. $y = \cos^{-1} \left(1 - \dfrac{x}{a}\right)$.

62. $y = \sin^{-1} (\cos x)$.

63. $y = \cos^{-1} \dfrac{1}{x}$.

64. $y = x \cos^{-1} x - \sqrt{1 - x^2}$.

65. $y = x \sin^{-1} x + \sqrt{1 - x^2}$.

66. $y = x \cos^{-1} 2x - \tfrac{1}{2}\sqrt{1 - 4x^2}$.

67. $y = a \sin^{-1} \dfrac{x}{a} + \sqrt{a^2 - x^2}$.

68. $y = \sin^{-1} \dfrac{x^2}{x^2 + 2}$.

69. $y = x^2\sqrt{1 - x^4} + \sin^{-1} x^2$.

70. $y = \tan^{-1} \dfrac{2x}{3}$.

71. $y = \tan^{-1} \dfrac{x - a}{x + a}$.

72. $y = \dfrac{2x}{x^2 + 4} + \tan^{-1} \dfrac{x}{2}$.

73. $y = \cot^{-1} \dfrac{1}{2}\left(\dfrac{a}{x} - \dfrac{x}{a}\right)$.

74. $y = \dfrac{ax}{x^2 + a^2} + \cot^{-1} \dfrac{a}{x}$.

75. $y = \tan^{-1} (\tan 2x)$.

76. $y = \tan^{-1} \dfrac{2x}{1 - x^2}$.

77. $y = \sqrt{x^2 - 1} - \tan^{-1}\sqrt{x^2 - 1}$.

78. $y = 3 \sec^{-1} (2x - 1)$.

79. $y = \sqrt{x^2 - a^2} - a \sec^{-1} \dfrac{x}{a}$.

80. $y = \sec^{-1} \sqrt{x}$.

81. $y = \csc^{-1} \dfrac{1}{x}$.

82. $y = \dfrac{\sqrt{x^2 - a^2}}{x^2} - \dfrac{1}{a} \csc^{-1} \dfrac{x}{a}$.

83. $y = \csc^{-1} \dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$.

84. $y = \tan^{-1} \dfrac{4 \sin x}{3 + 5 \cos x}$.

85. $y = \dfrac{1}{2} \tan^{-1} \left(\dfrac{1}{2} \tan \dfrac{x}{2}\right)$.

Determine the following integrals:

86. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}}$.

87. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$.

88. $\displaystyle\int \sec^2 x \, dx$.

89. $\displaystyle\int \sec x \tan x \, dx$.

90. If s is the arc from the x-axis to the point (x, y) on the circle $x^2 + y^2 = a^2$, show that

$$s = a \cos^{-1} \frac{x}{a}, \quad \frac{ds}{dx} = -\frac{a}{y}.$$

91. If A is the area bounded by the circle $x^2 + y^2 = a^2$, the y-axis, and the vertical line through (x, y), show that

$$A = xy + a^2 \sin^{-1} \frac{x}{a}, \quad \frac{dA}{dx} = 2y.$$

92. By interpreting the integral as an area show that

$$\int_0^x \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a}.$$

93. A tablet 7 feet high is placed on a wall with its base 9 feet above the level of an observer's eye. How far from the wall should the observer stand so that the angle of vision subtended by the tablet may be a maximum?

94. What is the angular velocity in radians per second of the second hand of a watch?

95. Assume that a 12-inch shell rotates once about its axis while moving forward 40 feet. If its speed is 2400 feet per second, find its angular velocity.

96. A wheel of radius 2 feet rolls down an inclined plane, its center moving the distance $s = 5t^2$ in t seconds. Find the angular velocity of the wheel about its axis.

97. The flywheel of a watch turns through the angle

$$\theta = A \sin \omega t$$

in t seconds. Find its angular velocity and acceleration at the instant when θ has its greatest value.

98. OA is a crank revolving with angular velocity ω about O. AB is a connecting rod attached to a piston B which moves along a straight line through O. Show that the speed of B is $\omega \cdot OC$, where C is the point in which the line BA intersects the line through O perpendicular to OB.

99. In a simple harmonic motion

$$s = A \sin (bt + c).$$

For what values of $bt + c$ is the velocity greatest?

100. At time t the displacement of a particle from its equilibrium position is

$$s = 5 \sin 6t.$$

Find the amplitude and period of vibration.

101. The displacement of a particle at time t is

$$s = 3 \cos 2t + 4 \sin 2t.$$

Show that its motion is simple harmonic, and find the amplitude and frequency.

102. A particle moves along the x-axis, its position at time t being

$$x = 2 - 3 \sin^2 2t.$$

Show that the motion is simple harmonic, and find its amplitude and period. To do this express $\sin^2 2t$ in terms of the angle $4t$.

103. A particle moves with simple harmonic motion along the x-axis, its displacement from the origin at time t satisfying the equation

$$\frac{d^2x}{dt^2} = -k^2x.$$

Find the period of vibration.

CHAPTER VII

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

83. Exponential and Logarithm. If a is positive and p, q are integers, the notation $a^{\frac{p}{q}}$ is used to represent the positive root

$$(a^p)^{\frac{1}{q}}.$$

When x is irrational a^x is defined as the limit

$$a^x = \lim_{r \to x} a^r$$

as r tends to x through rational values.

If a is a positive constant, not equal to 1,

$$y = a^x \tag{1}$$

is called an *exponential function* of x. When a is greater than 1, the function increases steadily as x increases. When a is less than 1,

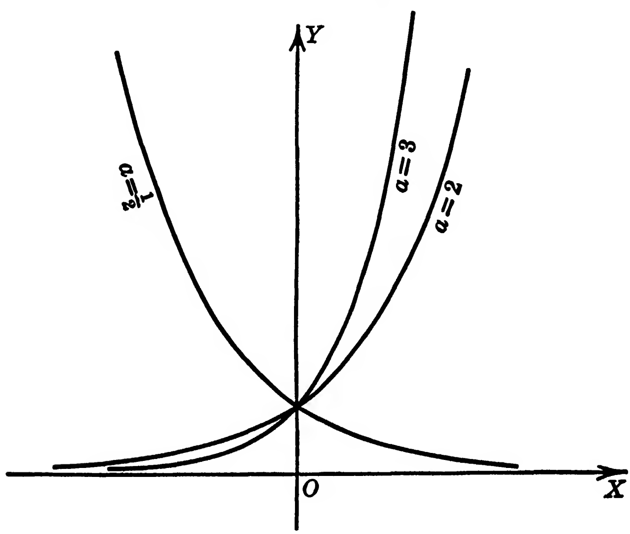

FIGURE 113.

it decreases as x increases. In either case y takes each positive value just once.

For positive values of y equation (1) therefore determines x as a function of y. This function, written

$$x = \log_a y, \tag{2}$$

is called the *logarithm of y to the base a.*

For real values of x and positive values of y equations (1) and (2) are by definition equivalent. Elimination of x gives the important identity

$$a^{\log_a y} = y \tag{3}$$

which states that the logarithm of a number is the power to which the base must be raised to equal the number.

It should be noted that logarithms are defined only for a positive base a. If a is negative a^x may not be real. Also logarithms of only positive numbers are here considered. Later we shall extend the definition to include negative numbers but will then find that the logarithm of a negative number is not real.

84. Limit of $(1 + h)^{\frac{1}{h}}$. In determining derivatives of logarithms we shall need the limit approached by the function

$$(1 + h)^{\frac{1}{h}}$$

when h tends to zero. In discussing this it will be convenient to distinguish three cases depending on the range of values h is allowed to take.

First suppose that

$$\frac{1}{h} = n$$

takes only positive integral values. Then, by use of the binomial theorem,

$$(1 + h)^{\frac{1}{h}} = \left(1 + \frac{1}{n}\right)^n \tag{1}$$

can be expressed as a sum of $n + 1$ terms

$$1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n^n}. \tag{2}$$

Each term in this series is positive and not greater than the corresponding term in the series

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots, \tag{3}$$

the convergence of which is discussed in Example 1, §145. Furthermore each term of (2) tends to the corresponding term of (3) as limit. By taking n large enough we can thus make the first $m + 1$ terms of (2) as nearly equal to the corresponding part of (3) as we please. For any fixed integer m we shall therefore have

$$e > \left(1 + \frac{1}{n}\right)^n > 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!}$$

when n is sufficiently large. Since m may be as large as you please this requires

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e. \tag{4}$$

Next suppose that h takes all positive values. For each value $h < 1$ there is a positive integer n such that

$$n < \frac{1}{h} \leqq n + 1,$$

whence

$$\left(1 + \frac{1}{n + 1}\right)^n < (1 + h)^{\frac{1}{h}} < \left(1 + \frac{1}{n}\right)^{n+1}.$$

This can be written

$$\frac{\left(1 + \dfrac{1}{n + 1}\right)^{n+1}}{1 + \dfrac{1}{n + 1}} < (1 + h)^{\frac{1}{h}} < \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)^n.$$

When h tends to zero, n tends to infinity and the first and third terms of this last inequality tend to e as limit. Since

$$(1 + h)^{\frac{1}{h}}$$

lies between the two, it must have the same limit.

Finally, suppose that

$$h = -k$$

is negative. Then

$$(1 + h)^{\frac{1}{h}} = (1 - k)^{-\frac{1}{k}} = \left(\frac{1}{1 - k}\right)^{\frac{1}{k}},$$

which is equivalent to

$$\left(1 + \frac{k}{1 - k}\right)^{\frac{1}{k}} = \left(1 + \frac{k}{1 - k}\right)\left(1 + \frac{k}{1 - k}\right)^{\frac{1-k}{k}}.$$

When h tends to zero

$$\frac{k}{1 - k}$$

tends to zero and the last expression again tends to e as limit.

Thus whether h varies through rational or irrational, positive or negative, values

$$\lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e. \tag{5}$$

By summing the series (3) the value of this constant to four places of decimals is found to be

$$e = 2.7183. \tag{6}$$

85. Differentiation of Logarithms. Let

$$y = \log_a u. \tag{1}$$

When u takes the increment Δu the corresponding increment in y is

$$\Delta y = \log_a (u + \Delta u) - \log_a u = \log_a \left(\frac{u + \Delta u}{u}\right) = \log_a \left(1 + \frac{\Delta u}{u}\right).$$

Thus

$$\frac{\Delta y}{\Delta u} = \frac{\log_a \left(1 + \dfrac{\Delta u}{u}\right)}{\Delta u}.$$

If we write

$$\frac{\Delta u}{u} = h,$$

this becomes

$$\frac{\Delta y}{\Delta u} = \frac{\log_a (1 + h)}{uh} = \frac{1}{u} \log_a (1 + h)^{\frac{1}{h}}. \tag{2}$$

As Δu tends to zero, h tends to zero and

$$(1 + h)^{\frac{1}{h}}$$

tends to the limit e [§84, (5)]. Thus (2) gives as limit

$$\frac{dy}{du} = \frac{\log_a e}{u}, \tag{3}$$

whence

$$dy = d(\log_a u) = \frac{\log_a e}{u}\, du. \tag{4}$$

Equation (4) is valid if a is any positive constant different from 1. In particular, if $a = e$,

$$\log_a e = \log_e e = 1$$

and

$$d(\log_e u) = \frac{du}{u}\cdot \tag{5}$$

Because of the greater simplicity of this formula logarithms to base e are called *natural* logarithms and are used almost exclusively in theoretical work.

The natural logarithm is indicated by the symbol ln. That is,

$$\ln u = \log_e u, \tag{6}$$

and equation (5) is usually written

$$d \ln u = \frac{du}{u}\cdot \tag{7}$$

86. Differentiation of Exponentials. To derive a formula for the differential of

$$y = a^u \tag{1}$$

write the equation in the inverse form

$$u = \log_a y. \tag{2}$$

By §85, (4) we then have

$$du = \frac{\log_a e}{y}\, dy,$$

whence

$$dy = \frac{1}{\log_a e}\, y\, du = \ln a \cdot y\, du.$$

Replacing y by a^u, we obtain

$$d(a^u) = a^u \ln a \, du. \tag{3}$$

In particular, if $a = e$, $\ln a = \ln e = 1$ and (3) becomes

$$d(e^u) = e^u \, du. \tag{4}$$

87. Differentials of Exponential and Logarithmic Functions. In the preceding sections we have proved the following formulas:

XIX. $d \log_a u = \dfrac{\log_a e}{u} du.$

XX. $d \ln u = \dfrac{du}{u}.$

XXI. $d(a^u) = a^u \ln a \, du.$

XXII. $d(e^u) = e^u \, du.$

Sometimes the calculation of a derivative is simplified if a logarithm is replaced by an equivalent form by means of the following formulas:

$$\ln (uv) = \ln u + \ln v. \tag{1}$$

$$\ln \frac{u}{v} = \ln u - \ln v. \tag{2}$$

$$\ln u^n = n \ln u. \tag{3}$$

$$\ln u^{\frac{1}{n}} = \frac{1}{n} \ln u. \tag{4}$$

Example 1. $y = \ln(x^2 + 1)$.
By formula XX,

$$\frac{dy}{dx} = \frac{\dfrac{d}{dx}(x^2 + 1)}{x^2 + 1} = \frac{2x}{x^2 + 1}.$$

Example 2. $y = \ln \sqrt{\dfrac{x - 1}{x + 1}}.$

By equations (4) and (2) this can be written

$$y = \tfrac{1}{2}[\ln(x - 1) - \ln(x + 1)],$$

whence

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x - 1} - \frac{1}{x + 1}\right] = \frac{1}{x^2 - 1}.$$

Example 3. $y = e^{\tan x}$.
By formula XXII

$$\frac{dy}{dx} = e^{\tan x}\frac{d}{dx}(\tan x) = e^{\tan x}\sec^2 x.$$

88. Logarithmic Differentiation. In some cases the derivative of a function

$$y = f(x)$$

can be found more easily by taking the natural logarithm of both sides, differentiating, and then solving for $\frac{dy}{dx}$. This process is called *logarithmic differentiation.*

Example. If u is positive and u and v are functions of x, find the derivative of u^v.

Writing

$$y = u^v$$

and taking logarithms of both sides, we have

$$\ln y = v \ln u,$$

whence by differentiation

$$\frac{1}{y}\frac{dy}{dx} = \ln u\frac{dv}{dx} + \frac{v}{u}\frac{du}{dx}.$$

Thus

$$\frac{dy}{dx} = y\left[\ln u\frac{dv}{dx} + \frac{v}{u}\frac{du}{dx}\right].$$

Replacing y by u^v, this becomes

$$\frac{d}{dx}(u^v) = u^v \ln u\frac{dv}{dx} + vu^{v-1}\frac{du}{dx}. \tag{1}$$

The function u^v is variable in two ways, namely through variation of u and through variation of v. If v alone were variable the derivative would be the first term in the above equations. If u alone were variable, the derivative would be the second term. When the two vary simultaneously the derivative is thus the sum of the results obtained by varying them one at a time.

89. Integration. Since an indefinite integral is merely a function with given differential, any differentiation formula gives by inversion a formula of integration. Thus from

$$d\ln u = \frac{du}{u}$$

we obtain

$$\int\frac{du}{u} = \ln u + C.$$

Each formula obtained in this way contains a single variable u and its differential. To integrate an expression which contains more than one variable it must be expressed as a sum of terms each of which contains only one variable and its differential. Reduction of an expression to such a form is called *separation of the variables.* If the variables cannot be thus separated, the expression cannot be integrated by our present methods.

In case of an equation, multiplication or division by a proper factor may separate the variables. In the equation

$$x\,dy - y\,dx = 0,$$

for example, the variables x and y are not separated, but division by xy gives

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

in which they are.

Example 1. Find the value of

$$\int \frac{dx}{2x+1}.$$

Taking

$$2x + 1 = u$$

we have

$$dx = \tfrac{1}{2}\,du$$

and

$$\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(2x+1) + C.$$

Example 2. Find the curves such that the part of the tangent included between the coördinate axes is bisected at the point of tangency.

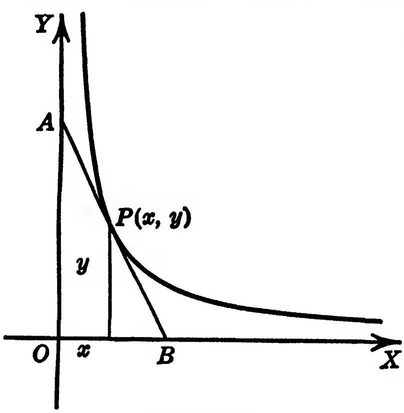

FIGURE 114.

In Figure 114 let AB be a tangent to the curve and $P(x, y)$ its point of tangency. Since P is the middle point of AB,

$$OA = 2y, \quad OB = 2x.$$

The slope of the curve at P is

$$\frac{dy}{dx} = -\frac{OA}{OB} = -\frac{y}{x}.$$

This can be written

$$\frac{dy}{y} + \frac{dx}{x} = 0.$$

Since each term contains a single variable, we can integrate and so get

$$\ln y + \ln x = C.$$

This is equivalent to

$$\ln xy = C.$$

Hence

$$xy = e^C = k.$$

C, and consequently k, can have any value. The curves are rectangular hyperbolas with the coördinate axes as asymptotes.

Example 3. A cylindrical tank with vertical axis and full of water has a leak at the bottom. Assuming that the water escapes at a rate proportional to the pressure at the leak and that 10% escapes during the first hour, how long will it take to half empty?

Let a be the radius of the tank, h its altitude, and x the depth of water at the end of t hours. The volume of water at time t is then $\pi a^2 x$, and its rate of outflow

$$-\pi a^2 \frac{dx}{dt}.$$

This is assumed to be proportional to the pressure at the bottom and so to x. That is

$$\pi a^2 \frac{dx}{dt} = kx,$$

where k is constant. Separating the variables, we have

$$\pi a^2 \frac{dx}{x} = k\,dt.$$

Integration gives

$$\pi a^2 \ln x = kt + C.$$

When $t = 0$ the tank is full and $x = h$. Hence

$$\pi a^2 \ln h = C.$$

Subtracting this from the preceding equation, we get

$$\pi a^2 \ln \frac{x}{h} = kt.$$

When $t = 1$, $x = \frac{9}{10}h$. Consequently

$$\pi a^2 \ln \tfrac{9}{10} = k.$$

When the tank is half empty, $x = \frac{1}{2}h$ and

$$t = \frac{\pi a^2 \ln \dfrac{x}{h}}{k} = \frac{\ln \frac{1}{2}}{\ln \frac{9}{10}} = 6.58 \text{ hr}.$$

PROBLEMS

1. By summing the series

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

determine the value of e correct to two decimal places.

2. By use of a logarithm table find the value of

$$(1 + \tfrac{1}{100})^{100},$$

and compare the result with e.

3. By writing

$$(1 + h)^{\frac{n}{h}} = [(1 + h)^{\frac{1}{h}}]^n$$

determine

$$\lim_{h \to 0} (1 + h)^{\frac{n}{h}}.$$

Plot the graphs of the following equations:

4. $y = \log_{10} x.$

5. $y = \ln x.$

6. $y = e^x.$

7. $y = e^{\frac{1}{x}}.$

8. $y = e^x \sin x.$

9. $y = \ln \dfrac{1 - x}{1 + x}.$

Find $\dfrac{dy}{dx}$ in each of the following:

10. $y = \ln (3x + 2).$

11. $y = \ln (3x^2 + 6x + 1).$

12. $y = \ln (x^2 - 4x + 4).$

13. $y = x \ln x - x.$

14. $y = \ln \dfrac{x - 1}{x + 1}.$

15. $y = \ln \sqrt{\dfrac{2 + 3x}{2 - 3x}}.$

16. $y = \ln \sin x.$

17. $y = \ln \tan \dfrac{x}{2}.$

18. $y = \ln (\sec 2x + \tan 2x).$

19. $y = \ln (x + \sqrt{x^2 - a^2}).$

20. $y = \ln \sin \dfrac{x}{3} + \dfrac{1}{2} \cos^2 \dfrac{x}{3}.$

21. $y = \ln \dfrac{1 - \cos 2x}{1 + \cos 2x}.$

22. $y = (\ln x)^2.$

23. $y = \log_{10} (x + 5).$

24. $y = \frac{1}{2}(e^x + e^{-x}).$

25. $y = e^{3x^2}.$

26. $y = \dfrac{2}{e^{3x}}.$

27. $y = e^{\sin x}.$

28. $y = x^e + e^x.$

29. $y = 10^x.$

30. $y = x^n n^x.$

31. $y = (3x + 1)e^{-3x}.$

32. $y = (2x^2 - 2x + 1)e^{2x}.$

33. $y = \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}.$

34. $y = \ln \dfrac{e^x - e^{-x}}{e^x + e^{-x}}.$

35. $y = e^{-3x} \cos 2x.$

36. $y = \frac{1}{5}e^x(\sin 2x - 2 \cos 2x).$

37. $y = x \sec^{-1} x - \ln (x + \sqrt{x^2 - 1}).$

38. $y = x \tan^{-1} \frac{x}{2} - \ln (x^2 + 4)$.

39. $y = \ln (\ln ax)$.

40. $2x + y = \ln (2x - y)$.

41. $e^x + e^y = 1$.

42. $\ln (xy) = x + y$.

43. $\tan y = e^x$.

44. $e^{x-y} = \ln \frac{x}{y}$.

45. $y = x^{-x}$.

46. $y = (\tan x)^x$.

47. $y = \sqrt{\frac{x^2 - a^2}{x^2 + a^2}}$.

48. $y = \frac{x\sqrt{2x + 3}}{\sqrt{3x + 2}}$.

49. $y = \frac{x^2\sqrt{x^2 - 4}}{\sqrt{x^2 + 4}}$.

50. By logarithmic differentiation prove the formula for the derivative of x^n when n is irrational.

51. Find the value of

$$\lim_{x \to 0} (1 + mx)^{\frac{n}{x}},$$

by taking the logarithm of

$$(1 + mx)^{\frac{n}{x}}$$

and finding the limit by l'Hospital's rule.

52. If $y = Ae^{nx} + Be^{-nx}$, where A, B, n are constants, show that

$$\frac{d^2y}{dx^2} - n^2y = 0.$$

53. If $y = ze^{-3x}$, where z is a function of x, show that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}\frac{d^2z}{dx^2}.$$

54. Find the slope of the curve $y = \ln (3x + 2)$ at the point where it crosses the x axis.

55. If $z = a^x$, find

$$\frac{d^nz}{dx^n}.$$

56. If $y = u^mv^n$, show that

$$\frac{dy}{y} = m\frac{du}{u} + n\frac{dv}{v}.$$

57. A box has a square base. By applying the formula of the preceding problem find the percentage change in the volume when the side of base is increased 2% and the altitude decreased 3%.

58. Find the maximum value of the function x^3e^{-x}.

59. On the part of the curve

$$y = e^x \sin x$$

between $x = \pi$ and $x = 2\pi$ find the point at greatest distance from the x-axis.

60. Find the least value of the function x^x for positive values of x.

61. If $0 < x < 1$ and

$$z = x \ln x - \frac{x^2}{2} + \frac{1}{2},$$

show that

$$\frac{d^2z}{dx^2} > 0.$$

From this and the fact that z and $\frac{dz}{dx}$ are zero at $x = 1$ show that $\frac{dz}{dx}$ is negative and z is positive. In this way show that

$$x \ln x > \frac{x^2}{2} - \frac{1}{2}$$

for values of x in the interval $0 < x < 1$.

62. By a method similar to that followed in the last problem, show that

$$\ln \sec x > \frac{x^2}{2}$$

for values of x in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

63. A particle moves along the x-axis, its abscissa at time t being

$$x = ae^{kt} + be^{-kt},$$

a, b, k being constants. Show that it is repelled from the origin with a force proportional to the distance.

Evaluate the following integrals:

64. $\displaystyle\int \frac{dx}{2x+3}$.

65. $\displaystyle\int \frac{x\,dx}{x^2+1}$.

66. $\displaystyle\int e^{3x}\,dx$.

67. $\displaystyle\int xe^{x^2}\,dx$.

68. Find the area bounded by the x-axis, the hyperbola $xy = 4$, and the ordinates at $x = 1$, $x = 4$.

69. The area bounded by the y-axis, the curve $x^2y = 4$, and the lines $y = 1$, $y = 4$ is rotated about the y-axis. Find the volume generated.

70. When bacteria grow without restraint they increase at a rate proportional to the number present. Express that number as a function of the time.

71. The portion of the tangent to a curve between the x-axis and the point of tangency is bisected by the y-axis. If the curve passes through (1, 2), find its equation.

72. A rotating wheel slows in such a way that its angular acceleration is proportional to its angular velocity. If the angular velocity is initially 200 revolutions per minute and in 2 minutes this slows to 50 revolutions per minute, when will it become 25 revolutions per minute?

CHAPTER VIII

PARAMETRIC EQUATIONS

90. Parameter. Sometimes the coördinates x, y of the variable point on a curve are expressed in terms of a third variable instead of in terms of each other. Such a third variable is called a *parameter*, and the equations expressing x, y in terms of the parameter are called *parametric equations* of the curve.

Thus, if t is the parameter and $f_1(t)$, $f_2(t)$ given functions, the locus of points

$$x = f_1(t), \quad y = f_2(t) \tag{1}$$

is a definite curve. To construct the curve we assign values to t, calculate the corresponding values of x and y, and plot the resulting points.

To obtain the Cartesian equation of the curve we eliminate the parameter. Suppose, for example, that by eliminating t from equations (1) we get

$$f(x, y) = 0. \tag{2}$$

This equation, being a consequence of the parametric equations, is satisfied by the coördinates of any point on the curve. If, conversely, for every pair of values x, y satisfying (2) a value t can be found such that $x = f_1(t)$, $y = f_2(t)$, then (2) is the Cartesian equation of the curve.

Example 1. Construct the graph represented by

$$x = t - \frac{1}{t}, \quad y = t + \frac{1}{t},$$

and find its Cartesian equation.

To construct the curve imagine t starting from $-\infty$ to increase steadily to $+\infty$. When t is negative and very large, x and y are nearly equal, y being slightly larger. The point (x, y) is then below and near the line $y = x$. As t increases to -1, (x, y) moves to A (Figure 115). As t continues to increase from -1 to zero, x becomes positive and (x, y) moves away, approaching the asymptote $y = -x$. As t passes through zero, the point (x, y) reappears in the second quadrant, crosses the x-axis at

$t = 1$, and then moves away, approaching the asymptote $y = x$ as t tends to infinity.

By squaring and subtracting we obtain

$$y^2 - x^2 = 4$$

the Cartesian equation. The curve is thus a rectangular hyperbola.

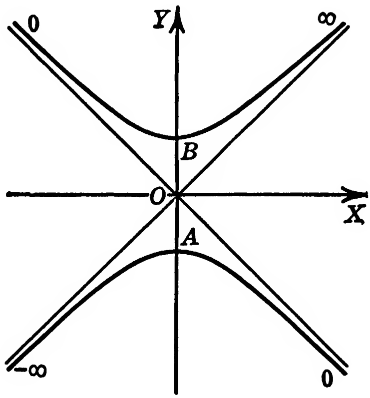

FIGURE 115.

Example 2. $x = \cos(2\phi)$, $y = \sin\phi$.

As ϕ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$ the point (x, y) traces the arc ABC (Figure 116). As ϕ continues to advance (x, y) oscillates back and forth along this path. The arc ABC is therefore the entire locus.

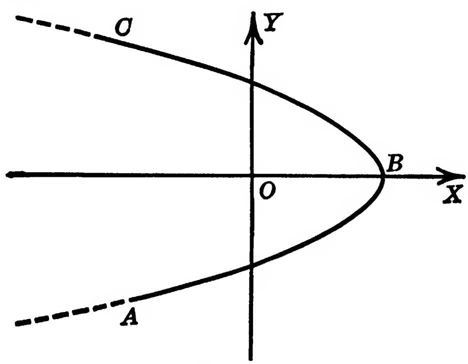

FIGURE 116.

Since $x = \cos 2\phi = 1 - 2\sin^2\phi$ and $y = \sin\phi$, elimination of ϕ gives

$$x = 1 - 2y^2.$$

The locus of this equation is not, however, entirely equivalent to that of

the parametric equations, for the Cartesian locus is an infinite parabola, whereas the parametric locus is only the arc ABC of that parabola.

Example 3. Determine parametric equations for the parabola

$$x^2 = 4py,$$

using as parameter the slope of the line from the origin to $P(x, y)$.

The parameter to be used is

$$t = \frac{y}{x}.$$

Solving this and the equation of the curve for x and y, we find

$$x = 4pt, \quad y = 4pt^2$$

as the parametric equations required.

91. The Cycloid. When a circle rolls along a straight line a point of its circumference describes a curve called a *cycloid*.

Let a circle of radius a and center C roll along the x-axis; let N be its point of contact with the x-axis and $P(x, y)$ a point of its

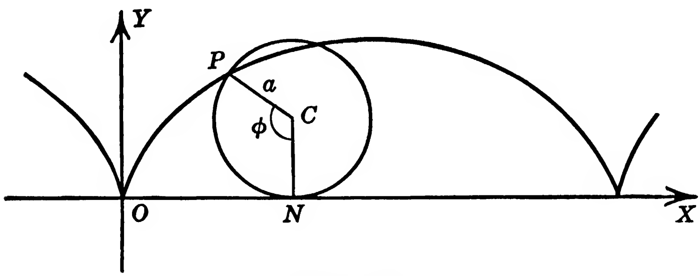

FIGURE 117.

circumference tracing a cycloid. Take as origin the point O found by rolling the circle to the left until P meets OX, and take as parameter the angle

$$\phi = NCP. \tag{1}$$

Since the point of contact moves the same distance along the circle and along the straight line,

$$ON = \text{arc } NP = a\phi. \tag{2}$$

The coördinates x, y of P are the horizontal and vertical displacements in moving from O to P, and these are the sums of corresponding displacements from O to C and from C to P. Thus

$$x = a\phi - a \sin \phi, \quad y = a - a \cos \phi \tag{3}$$

are parametric equations of the cycloid described by P.

92. The Epicycloid. When a circle rolls on the outside of a fixed circle, a point of its circumference describes a curve called an *epicycloid.*

Let a circle of radius a and center C roll on the outside of a fixed circle of radius b and center O; let N be the point of contact of the two circles, $P(x, y)$ a point describing an epicycloid, and A the point obtained by rolling the circle backward until P meets the fixed circle. Take the point O as origin, OA as x-axis, and take as parameter the angle $AOC = \phi$.

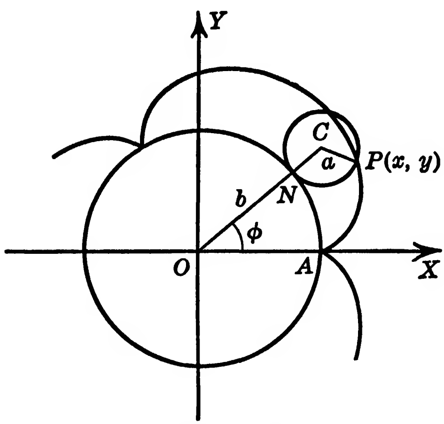

FIGURE 118.

Since the point of contact moves the same distance along both circles,

$$\text{arc } AN = \text{arc } NP.$$

If θ is the angle OCP we then have

$$a\theta = b\phi, \tag{1}$$

and so the angle between PC and the x-axis is

$$\theta + \phi = \frac{a+b}{a}\phi. \tag{2}$$

The coördinates of P are the horizontal and vertical displacements in moving from O to P. These displacements are the sums of corresponding displacements from O to C and from C to P. Since the displacements from C to P are the negatives of those from P to C, we thus have

$$\begin{aligned} x &= (a+b)\cos\phi - a\cos\left(\frac{a+b}{a}\phi\right), \\ y &= (a+b)\sin\phi - a\sin\left(\frac{a+b}{a}\phi\right) \end{aligned} \tag{3}$$

as parametric equations of the epicycloid.

93. Length of Arc. A parameter frequently used is the length of arc along a curve from a fixed to a variable point.

To define what we mean by the length of an arc AB divide it into parts by intermediate points $P_1, P_2, \cdots, P_{n-1}$ and join con-

secutive points by straight lines. The length of the resulting polygonal line from A to B is

$$L = AP_1 + P_1P_2 + \cdots + P_{n-1}B. \tag{1}$$

Now increase the number of points of division in such a way that the distances between consecutive points all tend to zero as n tends to infinity. If under these conditions L tends to a limit independent of the choice of intermediate points, the arc AB is said to be *rectifiable* and the limit is called its length.

Let $y = f(x)$ be the equation of the curve and assume

$$\frac{dy}{dx} = f'(x) \tag{2}$$

continuous on the arc AB. If $P(x, y)$, $Q(x + \Delta x, y + \Delta y)$ are consecutive vertices of the polygonal line, the chord between these points is

$$PQ = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x.$$

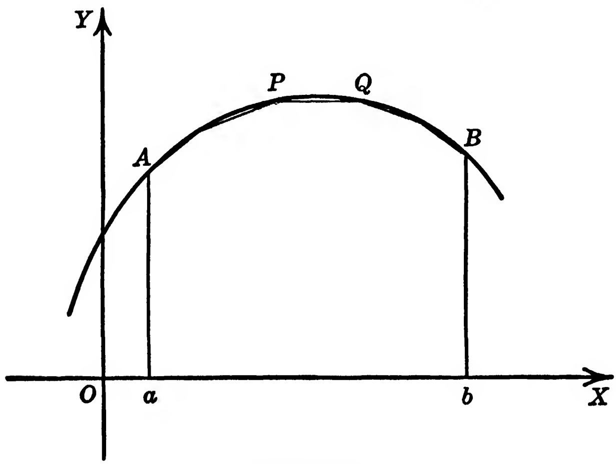

FIGURE 119.

Since the slope is continuous, by the mean value theorem there is some number ξ between x and $x + \Delta x$ such that

$$\frac{\Delta y}{\Delta x} = f'(\xi). \tag{3}$$

Thus

$$PQ = \sqrt{1 + [f'(\xi)]^2}\,\Delta x.$$

If a, b are the values of x at A, B and $b > a$, the total length of the polygonal line is

$$L = \sum_a^b \sqrt{1 + [f'(\xi)]^2}\, \Delta x. \tag{4}$$

When the number of divisions is increased and the increments Δx tend to zero this sum tends to

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

as limit. Thus, *if the slope* $\dfrac{dy}{dx}$ *is continuous at all points of an arc, the arc is rectifiable and its length from* $x = a$ *to* $x = b$ *is*

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{5}$$

An arc being given, the length L of the polygonal line, and so its limit s, is independent of the coördinates used. The integral (5) therefore depends only on the arc AB and not on the coördinate axes. If there is any set of coördinate axes with respect to which the slope is continuous, the arc is thus rectifiable and its length is given by (5) when referred to those axes. More generally, if an arc is the sum of a finite number of parts each of which is rectifiable, the whole arc is rectifiable and its length is the sum of the lengths of the parts.

94. Ratio of Arc and Chord. We are often concerned with the ratio of a small arc to the chord by which it is subtended. If the slope is continuous this ratio tends to the limit 1 when the chord tends to zero.

Since a sufficiently small arc with continuous slope is approximately straight, this may be taken as intuitively obvious. To prove it analytically let $P(x, y)$, $Q(x + \Delta x, y + \Delta y)$ be the ends of an arc and $y = f(x)$ the curve on which it is located. By the mean value theorems (§§34 and 50) there are values ξ_1, ξ_2 between x and $x + \Delta x$ such that

$$\text{chord } PQ = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x = \sqrt{1 + [f'(\xi_1)]^2}\, \Delta x,$$

$$\text{arc } PQ = \int_x^{x+\Delta x} \sqrt{1 + [f'(x)]^2}\, dx = \sqrt{1 + [f'(\xi_2)]^2}\, \Delta x.$$

Thus

$$\frac{\text{arc } PQ}{\text{chord } PQ} = \frac{\sqrt{1 + [f'(\xi_2)]^2}}{\sqrt{1 + [f'(\xi_1)]^2}}.$$

When Δx tends to zero, ξ_1 and ξ_2 tend to x, and this gives

$$\lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1. \tag{1}$$

95. Differential of Arc. Let $P_0(x_0, y_0)$ be a fixed point, $P(x, y)$ a variable point, on a given curve, and

$$s = P_0P \tag{1}$$

the arc from P_0 to P, this arc being considered positive for points of the curve on one side of P_0, negative for those on the other. If the slope is continuous at all points of the arc, by §93 (5)

$$s = \pm \int_{x_0}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{2}$$

the algebraic sign depending on the direction along the curve which has been taken as positive. This is a function of x with differential

$$ds = \pm \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \pm\sqrt{dx^2 + dy^2}. \tag{3}$$

Thus ds is the hypotenuse of a right triangle with sides dx and dy, as indicated in Figure 120. From this diagram it is also seen that

$$\frac{dx}{ds} = \cos\phi, \quad \frac{dy}{ds} = \sin\phi, \tag{4}$$

ϕ being the angle from the positive direction of the x-axis to the tangent drawn in the direction in which s increases.

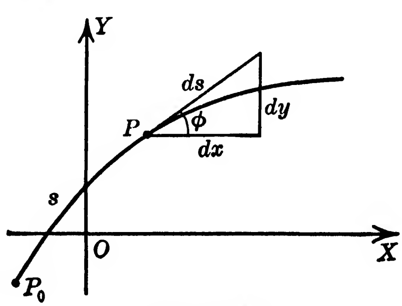

FIGURE 120.

When dx and dy are sufficiently small ds is approximately the arc between the points $P(x, y)$, $Q(x + \Delta x, y + \Delta y)$ on the curve. By treating this small arc as straight the above relations can be

read from a diagram of the type shown in Figure 121. Since the ratios of differentials are the limiting ratios of increments it is clear that on a curve with continuous slope this picture, although slightly inexact, will always suggest correct relations.

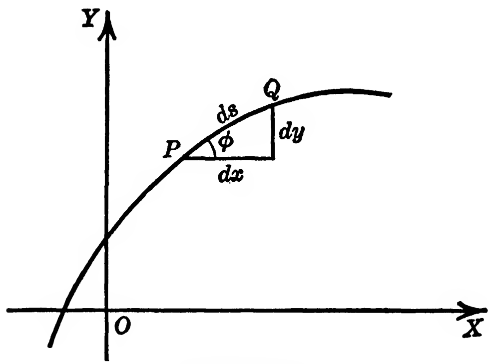

FIGURE 121.

Example. Find the arc of the curve

$$x = \cos\theta + \theta\sin\theta, \quad y = \sin\theta - \theta\cos\theta$$

from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$.

By differentiation we find

$$dx = \theta\cos\theta\, d\theta, \quad dy = \theta\sin\theta\, d\theta.$$

Therefore

$$ds = \sqrt{dx^2 + dy^2} = \theta\, d\theta.$$

Thus the required arc is

$$s = \int_0^{\frac{\pi}{2}} \theta\, d\theta = [\tfrac{1}{2}\theta^2]_0^{\frac{\pi}{2}} = \tfrac{1}{8}\pi^2.$$

96. Curvature. On a given curve the angle ϕ from the x-axis to the tangent at the variable point $P(x, y)$ may be regarded as a function of the arc s from a fixed point P_0 of the curve to P. As we move along the curve the change in ϕ measures the change in the direction of the tangent, and the derivative of ϕ with respect to s measures the change in direction per unit distance moved. This derivative

$$\frac{d\phi}{ds} \tag{1}$$

is called the *curvature* at P.

At a point where the curvature is large the change in direction per unit distance is large and the curve sharply bent. At a point where the curvature is small the change in direction and bending are more gradual.

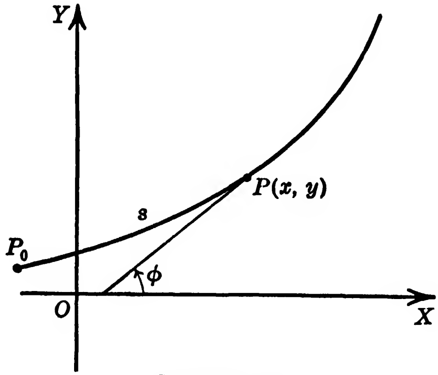

FIGURE 122.

To express the curvature in terms of x and y we use the equations

$$\phi = \tan^{-1}\frac{dy}{dx}, \quad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,$$

whence

$$\text{curvature} = \frac{d\phi}{ds} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{2}$$

The radical in the denominator is usually taken positive. This amounts to taking the curvature positive when the curve is concave upward, negative when concave downward.

In a circle of radius r (Figure 123)

$$\phi = \theta + \frac{\pi}{2},$$

and so the curvature is

$$\frac{d\phi}{ds} = \frac{d\theta}{ds} = \frac{1}{r}.$$

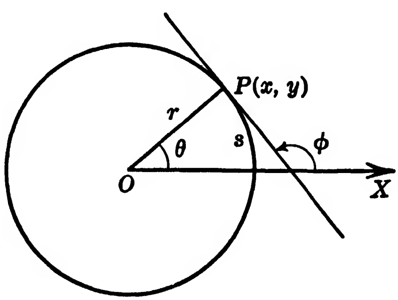

FIGURE 123.

Thus the reciprocal of curvature

$$\frac{ds}{d\phi}$$

is equal to the radius of the circle.

Similarly, the reciprocal of curvature

$$\rho = \frac{ds}{d\phi} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} \tag{3}$$

on any curve is called its *radius of curvature.* At a given point this is the radius of the circle which passes through the point and which has at that point the same slope and second derivative as the given curve. It is therefore the radius of the circle that most closely fits the curve in the immediate neighborhood of the point.

In the determination of curvature it is clear that the angle ϕ, instead of being measured from the x-axis, might be measured from any fixed direction. If we measure it from the y-axis, the coördinates x, y will be interchanged in the above discussion, giving

$$\rho = \frac{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}} \tag{4}$$

as the radius of curvature. If the radicals are taken positive in both cases the expressions (3) and (4) may, however, have different algebraic signs.

Example 1. Find the radius of curvature of the parabola

$$y^2 = 4px$$

at the origin.

Since

$$\frac{dy}{dx}$$

is not continuous at the origin the conditions assumed in deriving equa-

tion (3) are not satisfied. If, however, we measure slope and angle from the y-axis, we have

$$\frac{dx}{dy} = \frac{y}{2p} = 0, \quad \frac{d^2x}{dy^2} = \frac{1}{2p},$$

and

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} = 2p$$

as the radius of curvature at the origin.

Example 2. The curve formed by a flexible chain when held by its ends and hanging under its own weight is called a *catenary*. With suitable choice of axes its equation is

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

where a is constant. Find its radius of curvature.

By differentiation and substitution we find

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right),$$

$$\frac{d^2y}{dx^2} = \frac{1}{2a}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

The radius of curvature of the catenary is therefore

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{a}{4}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2 = \frac{y^2}{a}.$$

PROBLEMS

Plot the loci represented by the following pairs of parametric equations, and determine the corresponding Cartesian equations:

1. $x = 2 + 3t,\ y = 1 - 2t.$

2. $x = t^2,\ y = t + 2.$

3. $x = t^2 + 2t,\ y = t^2 - 2t.$

4. $x = 2\frac{1 - t^2}{1 + t^2},\ y = \frac{2t}{1 + t^2}.$

5. $x = 2\frac{1 + t^2}{1 - t^2},\ y = \frac{2t}{1 - t^2}.$

6. $x = \dfrac{4}{1 + t^2}$, $y = \dfrac{4}{t(1 + t^2)}$.

7. $x = t^2(t - 1)$, $y = t^2(t + 1)$.

8. $x = \sin \phi$, $y = \sin \left(\phi + \dfrac{\pi}{4}\right)$.

9. $x = \sec \phi$, $y = \tan \phi$.

10. $x = a \cos \phi$, $y = b \sin \phi$.

11. $x = \sin t$, $y = \sin 2t$.

12. $x = \cos t$, $y = \sec t$.

13. $x = 2 + 3 \sin t$, $y = 3 - 2 \sin t$.

14. $x = e^t \sin t$, $y = e^t \cos t$.

15. Are $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$, and $x = e^t + e^{-t}$, $y = e^t - e^{-t}$, parametric equations of the same locus?

16. Plot two turns of the curve

$$x = \cos \phi, \quad y = \sin \left(\frac{7\pi\phi}{22}\right).$$

Find the intersections of the following pairs of curves:

17. $\begin{cases} x = 3t^2, \\ y = 4t, \end{cases}$ $\begin{cases} x = 5 \cos \theta, \\ y = 5 \sin \theta. \end{cases}$ **18.** $\begin{cases} x = 2 + t, \\ y = 2 + \dfrac{1}{t}, \end{cases}$ $\begin{cases} x = 1 + 2t, \\ y = 1 - t. \end{cases}$

19. A circle of radius a has its center at the origin. Find parametric equations of the circle, using as parameter the angle ϕ from the x-axis to the radius through $P(x, y)$.

20. Find parametric equations of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

using the parameter ϕ defined by the equation $x = a \cos \phi$.

21. Find parametric equations of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

using the parameter ϕ defined by the equation $x = a \sec \phi$.

22. Find parametric equations of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

using the parameter defined by the equation $x = a \cos^3 \phi$.

23. Find parametric equations of the curve

$$x^3 + y^3 = xy,$$

using as parameter the slope of the line through the origin and the point (x, y).

24. Find parametric equations of the curve

$$y^2 = \frac{x^3}{2a - x},$$

using as parameter the slope of the line through the origin and the point (x, y).

Find parametric equations of the following curves, using as parameter the slope m of the curve at the point (x, y):

25. $y^2 = 4px$.

26. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

27. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

28. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

29. Find the slope of the curve

$$x = \frac{1}{(1 - t^2)^{\frac{3}{2}}}, \quad y = \frac{t^3}{(1 - t^2)^{\frac{3}{2}}}$$

at the point (x, y).

30. Find the angle from the x-axis to the line tangent to the curve

$$x = a \cos t(1 - \cos t), \quad y = a \sin t(1 + \cos t)$$

at the point (x, y).

31. A rod AB moves with its end A in the x-axis and B in the y-axis. Using as parameter the angle between the rod and the x-axis, find parametric equations of the locus described by the point $P(x, y)$ on the rod, if a and b are its distances from A and B. Also find the Cartesian equation of the locus.

32. A string held taut is unwound from a circle. Taking the origin at the center of the circle, the x-axis through the point where the string begins to unwind, and using as parameter the angle from the positive end of the x-axis to the radius through the point of tangency, determine parametric equations of the locus described by the end of the string. The locus is called an *involute* of the circle.

33. When a wheel rolls along a straight line the curve described by any point of a spoke is called a *trochoid.* Let the wheel roll along the x-axis and use as parameter the angle ϕ in the equation of the cycloid (§91). Find parametric equations of the trochoid described by the point (x, y) at distance b from the center of the wheel.

34. A *hypocycloid* is the locus described by a point on the circumference of a circle which rolls internally along the circumference of a fixed circle. Find parametric equations of the hypocycloid when the radius of the moving circle is one-fourth that of the fixed circle, using a parameter analogous to that in the equations of the epicycloid. Also find the Cartesian equation of the curve.

35. OA is diameter of a circle of radius a, AN is tangent, and ON intersects the circle at M. Lines through M and N, parallel respectively to AN and OA, intersect in P. Taking O as origin, OA as y-axis, and the angle AON as parameter, find parametric equations for the locus of P. Also find the Cartesian equation. The curve is called a *witch.*

36. Using the equations of the preceding problem, determine the area bounded by the witch and the x-axis.

37. OA is diameter of a circle of radius a, AN is tangent, and ON intersects the circle at M. On ON the point P is found such that $OP = MN$. Taking O as origin, OA as x-axis, and the angle AON as parameter, find parametric equations of the locus described by P. Also find the Cartesian equation. The curve is called a *cissoid.*

38. Through the point $A(-a, 0)$ on the x-axis a line is drawn intersecting the y-axis at B, and on this line two points $P(x, y)$ are taken at distance OB

from B. Find parametric equations for the locus of P, using as parameter the angle from the x-axis to the line AB. Also find the Cartesian equation of the curve. The locus is called a *strophoid*.

39. The angles of a triangle are A, B, C and the opposite sides a, b, c. If the vertex A moves along the x-axis and B along the y-axis, find the locus of C, using the angle from the x-axis to AB as parameter. If C is a right angle, show that the locus is a straight line.

40. Show that the normal to the cycloid at the point P (Figure 117) passes through the point N where the rolling circle touches the fixed line, and that the tangent intersects the circle at the upper end of the diameter through N.

41. Show that the normal to the epicycloid at the point P (Figure 118) intersects the fixed circle at N, and that the tangent intersects the rolling circle at the end of the diameter through N.

Find the arc lengths of the following curves between the points indicated:

42. $9y^2 = 4(x-1)^3$, $(1, 0)$ to $(2, \frac{2}{3})$.

43. $x^2 = 4y^3$, $y = 0$ to $y = 7$.

44. $x = a\cos\theta$, $y = a\sin\theta$, $\theta = 0$ to $\theta = 2\pi$.

45. $x = a\cos^3\phi$, $y = a\sin^3\phi$, $\phi = 0$ to $\phi = \frac{\pi}{2}$.

46. $x = e^t\sin t$, $y = e^t\cos t$, $t = 0$ to $t = \pi$.

47. $x = \sqrt{t^2+1}$, $y = \ln(t + \sqrt{t^2+1})$, $t = 0$ to $t = 5$.

48. $x = a\sec t$, $y = a\ln(\sec t - \tan t)$, $t = -\frac{\pi}{4}$ to $t = \frac{\pi}{4}$.

49. $x = 2\sin^2 t$, $y = 2t - \sin 2t$, $t = 0$ to $t = \pi$.

50. The length of arc on a certain curve from a fixed point to the variable point (x, y) is $s = x^2 - \frac{1}{4}$. Find the slope of the curve at $x = 2$.

51. If s is arc length on a cycloid, show that

$$ds^2 = 2a^2(1-\cos\phi)\,d\phi^2 = 4a^2\sin^2\frac{\phi}{2}\,d\phi^2.$$

Find the length of one arch of the cycloid.

52. If s is arc length on the epicycloid, show that

$$ds^2 = 2(a+b)^2\left(1-\cos\frac{b}{a}\phi\right)d\phi^2 = 4(a+b)^2\sin^2\frac{b\phi}{2a}\,d\phi^2.$$

Find the length of one arch of the epicycloid.

53. If the arc s of the catenary is measured from the lowest point, show that

$$\frac{dy}{dx} = \frac{s}{a}.$$

54. Show that the area bounded by the x-axis, an arc of the catenary, and two ordinates is proportional to the length of arc.

Find the radius of curvature on each of the following curves:

55. $y = \ln\sec x$.

56. $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$.

57. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

58. $x = a\cos^3 t$, $y = a\sin^3 t$.

59. $x = 2\sin^2 t$, $y = 2t - \sin 2t$.

60. $x = a\sec t$, $y = a\ln(\sec t - \tan t)$.

61. Find the radius of curvature of

$$y^2 = (x - 1)^3$$

at (2, 1).

62. Find the radius of curvature of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points where it crosses the x-axis.

63. Find the radius of curvature of

$$y = e^{-x}\sin 2x$$

at the origin.

64. Find the greatest and the least radii of curvature on the curve $y = \sin x$.

65. Find the radius of curvature of the epicycloid.

66. On an arch of the cycloid show that

$$\rho^2 + s^2 = 16a^2,$$

s being measured from the highest point of the arch.

CHAPTER IX

POLAR COÖRDINATES

97. Polar Coördinates. The most natural way to indicate the position of a point P in a plane is to give the distance and direction of P from a fixed point of the plane. Let O be the fixed point, called the origin, or *pole,* and OX a fixed line, called the *initial line,* or axis. The distance $r \doteq OP$ and the angle θ from OX to OP are called the *polar coördinates of P.*

The angle θ is any angle from OX to the line OP, this angle being considered positive when measured in the counterclockwise direction, negative when in the clockwise direction.

The radius r is considered positive when OP is the terminal side of θ (Figure 124, 1 and 3), and negative when θ terminates on OP produced (Figure 124, 2).

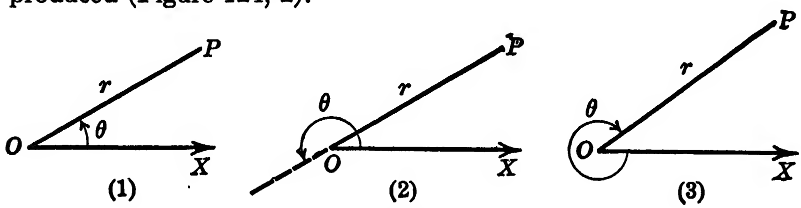

FIGURE 124.

A given pair of polar coördinates r, θ determines a definite point. But, since θ may be positive or negative and may wind any number of times around the origin, the same point is represented by infinitely many pairs of polar coördinates. It is mainly because of this multiple representation that polar coördinates are a little less simple than rectangular.

The point with polar coördinates r, θ is represented by the notation (r, θ). To indicate that the point P has the polar coördinates r, θ the notation $P(r, \theta)$ is used.

Example. Plot the point $P(-1, -\frac{1}{4}\pi)$ and find its other pairs of polar coördinates.

The point is shown in Figure 125. Since $r = -1$, the angle $\theta = -\frac{1}{4}\pi$ terminates on OP produced. The angle XOP is $\frac{3}{4}\pi$. Any other angle

terminating on OP or OP produced differs from one of these by a multiple of 2π. Any such angle has one of the forms $2n\pi - \frac{1}{4}\pi$ or $2n\pi + \frac{3}{4}\pi$, where n is a positive or negative integer. Consequently any polar coördinates of P have one of the forms

$$\left(-1, 2n\pi - \frac{\pi}{4}\right), \quad (1, 2n\pi + \tfrac{3}{4}\pi).$$

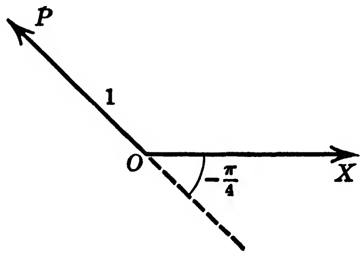

FIGURE 125.

98. Graphs. The locus of points with polar coördinates which satisfy a given equation

$$r = f(\theta)$$

is a curve. To plot the locus make a table of pairs of values r, θ satisfying the equation, plot the corresponding points, and draw a smooth curve through them.

It may be convenient to look for any of the things mentioned in connection with plotting in rectangular coördinates (§69). It is usually sufficient, however, to imagine θ to increase from some definite value and determine at each point merely whether r is increasing or decreasing, draw a curve on which r varies in the proper direction, and mark accurately the points where r is a maximum, a minimum, or zero. Proceed in a similar way with θ decreasing from the initial value.

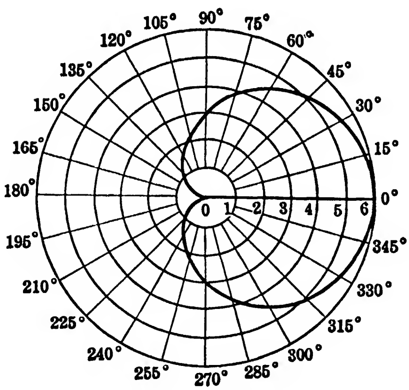

FIGURE 126.

The work is facilitated by using paper ruled as in Figure 126. In the following diagrams parts of the curves where r is negative are dotted.

Example 1. $r = a(1 + \cos\theta)$.

The curve called a *cardioid* is shown in Figure 126. As θ increases from 0 to π, r steadily diminishes from $2a$ to 0. As θ continues to increase, the cosine, and so r, retraces its values in inverse order, again reaching the maximum, $\cos\theta = 1$, $r = 2a$, at $\theta = 2\pi$. Because of the periodicity of $\cos\theta$, negative values of θ and values greater than 2π give points already plotted.

Example 2. $r = a \sin \frac{1}{2}\theta$.

As θ increases from 0, r increases reaching its maximum $r = a$ at

$$\frac{1}{2}\theta = \frac{\pi}{2}, \quad \theta = \pi.$$

As θ continues to increase, r retraces its values, returning to zero at $\theta = 2\pi$.

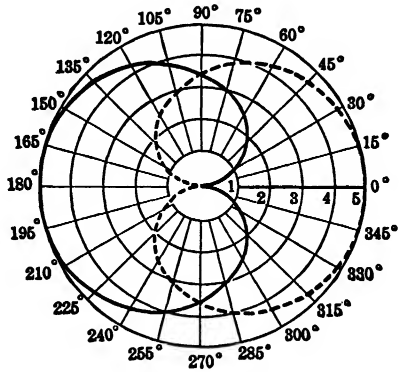

FIGURE 127.

Beyond $\theta = 2\pi$, r becomes negative, following the dotted portion of the curve (Figure 127). Since

$$\sin \tfrac{1}{2}(4\pi + \theta) = \sin (2\pi + \tfrac{1}{2}\theta) = \sin \tfrac{1}{2}\theta,$$

angles beyond $\theta = 4\pi$ and negative angles give points already plotted.

Example 3. $r = a\theta$.

The curve, called a *spiral of Archimedes*, is shown in Figure 128. Assuming that a is positive, as θ increases from zero r is positive and steadily

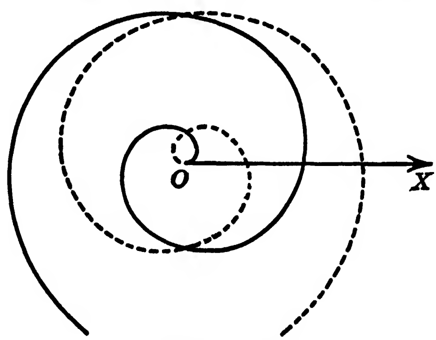

FIGURE 128.

increases. When the angle makes a complete turn about the origin, r is increased by the amount $2\pi a$. This part of the curve thus consists of a

set of expanding coils each at distance $2\pi a$ outside the preceding. Negative values of θ give a similar set of coils crossing the former on the vertical axis.

Example 4. $r^2 = 2a^2 \cos 2\theta$. To each value of θ correspond two values

$$r = \pm a\sqrt{2 \cos 2\theta}$$

differing only in algebraic sign. The curve is therefore symmetrical with respect to the origin. Also angles differing only in sign determine the same

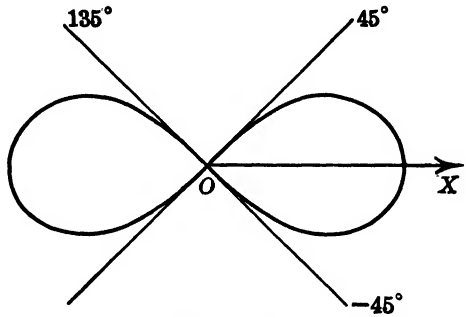

FIGURE 129.

value of r. Hence the curve is symmetrical with respect to the initial line. As θ varies from 0 to $\pm\frac{1}{4}\pi$, r varies from $\pm a\sqrt{2}$ to 0. When θ is between $\pm\frac{1}{4}\pi$ and $\pm\frac{3}{4}\pi$, $\cos 2\theta$ is negative and r imaginary. Other values of θ give points already plotted. The curve, called a *lemniscate*, is shown in Figure 129.

99. Intersection of Curves. If the polar coördinates of a point satisfy the equation of a curve, the point lies on the curve. A point may, however, be on a curve although its coördinates (as given) do not satisfy the equation. This happens because a point has many pairs of polar coördinates. One of these pairs may satisfy a given equation while another does not. Thus the point $B\left(a, -\frac{\pi}{2}\right)$ is on the curve $r^2 = a^2 \sin \theta$ (Figure 130) although the coördinates given do not satisfy the equation, for the coördinates $-a, \frac{\pi}{2}$ represent the same point and do satisfy the equation.

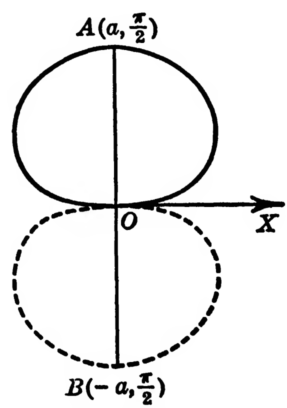

FIGURE 130.

To find the intersections of two curves we solve their equations simultaneously. The pairs of coördinates thus obtained represent points on both curves. There may, however, be other intersections. This happens when some of the pairs of polar coördinates representing a point satisfy one equation, other pairs satisfy the other equation, but no pair satisfies both. In finding intersections of loci represented by polar equations, the graphs should therefore be constructed. Any extra intersections will then be seen.

Example. Find the intersections of the curves

$$r^2 = a^2 \sin\theta, \quad r = a.$$

Solving simultaneously we get $\sin\theta = 1$, $\theta = \frac{\pi}{2}$. One point of intersection is therefore $A\left(a, \frac{\pi}{2}\right)$. From Figure 130 it is seen that $B\left(-a, \frac{\pi}{2}\right)$ is also an intersection.

100. Change of Coördinates. The same point may be represented by rectangular or by polar coördinates. It is sometimes

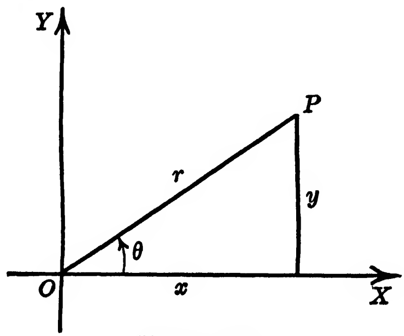

FIGURE 131.

desirable to use the two systems simultaneously. The x-axis is then usually coincident with the initial line, and the origin of rectangular coördinates is the pole. A point P (Figure 131) then has four coördinates x, y, r, θ. These are connected by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r = \pm\sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}.$$

By means of these equations (or, better, by direct use of a diagram) any expression in rectangular coördinates can be trans-

formed into a corresponding expression in polar coördinates, and conversely.

Example. By changing to rectangular coördinates determine the locus represented by the equation

$$r(2\cos\theta + 3\sin\theta) = 4.$$

Since

$$r\cos\theta = x, \quad r\sin\theta = y$$

the corresponding equation in rectangular coördinates is

$$2x + 3y = 4.$$

The locus is therefore a straight line.

101. Equations of Certain Curves. By transformation of coördinates, polar equations can be obtained for the curves we have represented in rectangular coördinates. In this section we determine directly some of these equations.

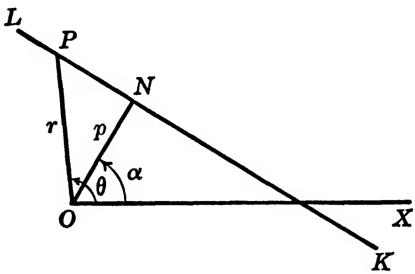

FIGURE 132.

(1) *Polar Equations of a Straight Line.* Let LK (Figure 132) be the straight line and ON the perpendicular upon it from the origin; let

$$ON = p, \quad XON = \alpha.$$

If $P(r, \theta)$ is any point on the line, ON is the base of a right triangle with hypotenuse $OP = r$ and base angle $NOP = \theta - \alpha$. Hence

$$r\cos(\theta - \alpha) = p \tag{1}$$

is the equation required.

(2) *Polar Equation of a Circle.* If the circle passes through the origin (Figure 133), let a be its radius and α the angle from the initial line to the diameter OA through the origin. If $P(r, \theta)$ is any point on the circle, OAP is a right triangle of hypotenuse

$OA = 2a$, side $r = OP$, and acute angle $AOP = \theta - \alpha$. Thus

$$r = 2a \cos (\theta - \alpha) \tag{2}$$

is the equation required.

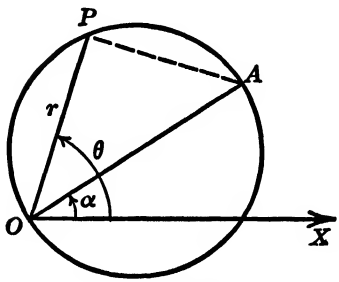

FIGURE 133.

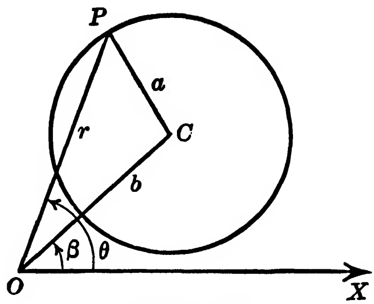

FIGURE 134.

If the circle does not pass through the origin (Figure 134) let its radius be a and its center $C(b, \beta)$. The triangle OCP has sides a, b, r and angle $\theta - \beta$ opposite a. Thus

$$a^2 = b^2 + r^2 - 2br \cos (\theta - \beta) \tag{3}$$

is the required equation.

(3) *Polar Equation of a Conic.* Let F be a focus and RS the corresponding directrix of an ellipse, parabola, or hyperbola (Figure 135). Take F as origin and the axis through F as initial line. If $DF = k$ is the distance from the directrix to the focus and $P(r, \theta)$ is any point on the curve,

$$FP = r, \quad NP = k + r \cos \theta.$$

In Chapter IV it was shown that FP is e times NP, where e is the eccentricity. Thus

$$r = e(k + r \cos \theta),$$

whence

$$r = \frac{ke}{1 - e \cos \theta} \tag{4}$$

is the equation of the curve.

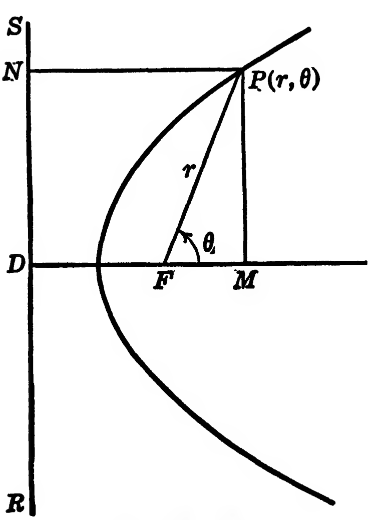

FIGURE 135.

102. Differential Relations. Let O be the origin of polar coördinates, $P(r, \theta)$ a point on a given curve, and s the arc length measured along the curve from a fixed point P_0 to P. The letter ψ is used to represent the angle from the direction along OP in which r increases to the tangent PT drawn in the direction in which s increases.

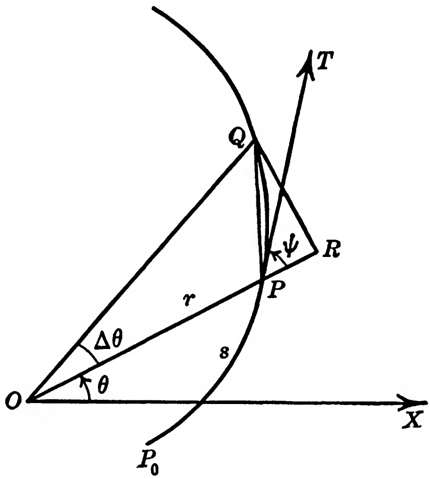

FIGURE 136.

To determine this angle let $Q(r + \Delta r, \theta + \Delta\theta)$ be a neighboring point on the curve (Figure 136). Draw QR perpendicular to PR, and let $\Delta s = \text{arc } PQ$. Then

$$\sin (RPQ) = \frac{RQ}{PQ} = \frac{(r + \Delta r) \sin \Delta\theta}{PQ} = (r + \Delta r) \frac{\sin \Delta\theta}{\Delta\theta} \frac{\Delta\theta}{\Delta s} \frac{\Delta s}{PQ},$$

$$\cos (RPQ) = \frac{PR}{PQ} = \frac{(r + \Delta r) \cos \Delta\theta - r}{PQ}$$

$$= \cos \Delta\theta \frac{\Delta r}{\Delta s} \frac{\Delta s}{PQ} - \frac{r(1 - \cos \Delta\theta)}{\Delta\theta} \frac{\Delta\theta}{\Delta s} \frac{\Delta s}{PQ}.$$

As $\Delta\theta$ tends to zero

$$\lim (RPQ) = \psi, \lim \frac{\sin \Delta\theta}{\Delta\theta} = 1, \lim \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0, \lim \frac{\Delta s}{PQ} = 1.$$

The above equations therefore give as limits

$$\sin \psi = \frac{r\, d\theta}{ds}, \quad \cos \psi = \frac{dr}{ds}. \tag{1}$$

These equations show that dr and $r\,d\theta$ are the sides of a right triangle with hypotenuse ds and base angle ψ (Figure 137). From

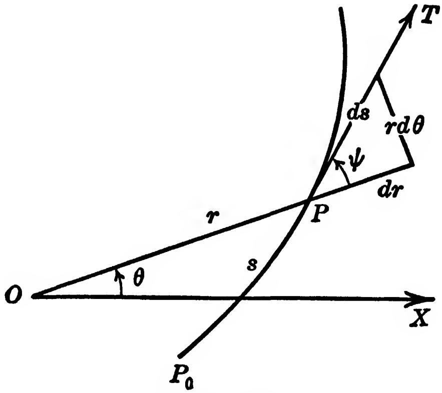

FIGURE 137.

this triangle all the relations connecting dr, ds, $d\theta$, and ψ can be obtained. In addition to (1) the most important of these are

$$\tan\psi = \frac{r\,d\theta}{dr} = \frac{r}{\dfrac{dr}{d\theta}}, \tag{2}$$

$$ds^2 = dr^2 + r^2\,d\theta^2. \tag{3}$$

The arc s is usually drawn along the curve in the direction in which θ increases. This will be assumed in the problems.

Example 1. The logarithmic spiral

$$r = ae^{k\theta}.$$

The curve is shown in Figure 138. In this case

$$dr = kae^{k\theta}\,d\theta$$

and

$$\tan\psi = \frac{r\,d\theta}{dr} = \frac{1}{k}.$$

The angle ψ between the radius and tangent is therefore constant. The equation

$$\frac{dr}{ds} = \cos\psi$$

shows that $\dfrac{dr}{ds}$ is also constant and so r and s increase proportionally.

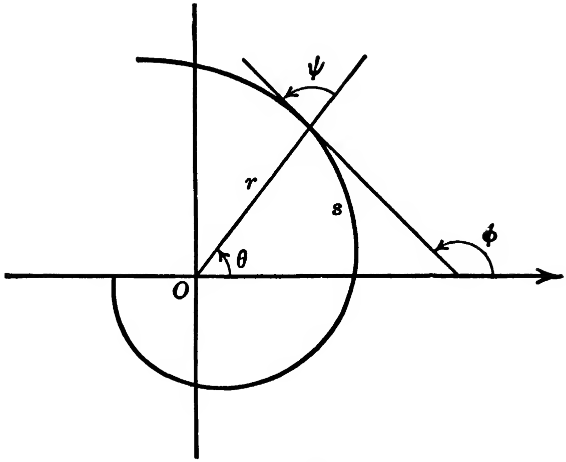

FIGURE 138.

Example 2. Find the slope of the logarithmic spiral at the point (r, θ). From the diagram (Figure 138) we have

$$\phi = \psi + \theta.$$

The slope is therefore

$$\tan\phi = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\tan\theta} = \frac{\dfrac{1}{k} + \tan\theta}{1 - \dfrac{1}{k}\tan\theta} = \frac{1 + k\tan\theta}{k - \tan\theta}.$$

At $\theta = 60°$, for example, the slope is

$$m = \frac{1 + k\sqrt{3}}{k - \sqrt{3}}.$$

Example 3. Find the angle between the cardioid $r = a(1 + \sin\theta)$ and the circle $r = 3a\sin\theta$ at each intersection.

The curves are shown in Figure 139. To find intersections we solve simultaneously, thus obtaining

$$1 + \sin\theta = 3\sin\theta,$$

whence $\sin\theta = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$. If ψ_1 and ψ_2 are the angles on the circle and cardioid respectively, we have at $\theta = \dfrac{\pi}{6}$

$$\tan\psi_1 = \frac{r_1}{\dfrac{dr_1}{d\theta}} = \frac{3a\sin\theta}{3a\cos\theta} = \frac{1}{\sqrt{3}},$$

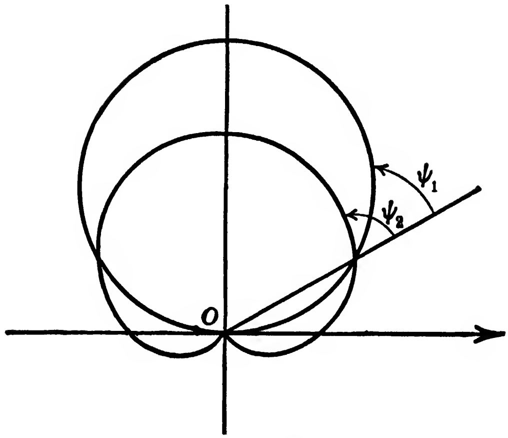

FIGURE 139.

$$\tan \psi_2 = \frac{r_2}{\frac{dr_2}{d\theta}} = \frac{a(1 + \sin \theta)}{a \cos \theta} = \sqrt{3}.$$

The angle

$$\beta = \psi_2 - \psi_1$$

from the circle to the cardioid thus satisfies the equation

$$\tan \beta = \frac{\tan \psi_2 - \tan \psi_1}{1 + \tan \psi_1 \tan \psi_2} = \frac{1}{\sqrt{3}},$$

whence $\beta = 30°$. By symmetry the angle at $\theta = \frac{5\pi}{6}$ has the same magnitude. At the origin the cardioid is perpendicular to the x-axis and the circle is tangent. At this point the angle between the curves is thus 90°.

103. Area in Polar Coördinates. We shall determine the area of a sector AOB bounded by two radii $\theta = \alpha$, $\theta = \beta > \alpha$, and the arc AB of a continuous curve

$$r = f(\theta).$$

Areas of a more general form can be expressed as sums or differences of areas of this kind.

Divide the angle AOB into parts by intermediate values θ_1, $\theta_2 \cdots, \theta_{n-1}$ arranged in order from $\theta = \alpha$ to $\theta = \beta$. In each interval θ, $\theta + \Delta\theta$ choose a value θ' and construct a circular sector of angle $\Delta\theta$ and radius $r' = f(\theta')$. The sum of the areas of these circular sectors is

$$\sum_{\alpha}^{\beta} \tfrac{1}{2}(r')^2 \, \Delta\theta.$$

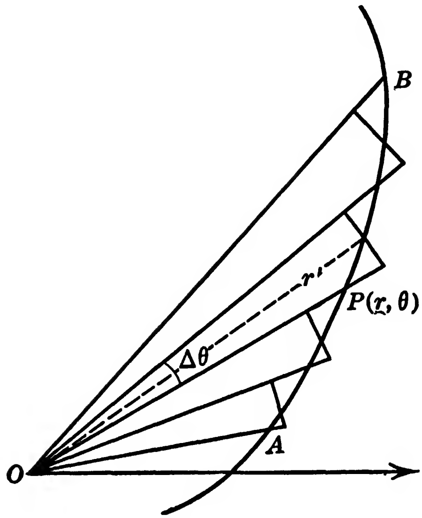

FIGURE 140.

When the number of divisions increases and the intervals $\Delta\theta$ all tend to zero this sum approaches the area AOB as limit. Therefore

$$A = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

is the area bounded by the curve and the radii $\theta = \alpha$, $\theta = \beta$.

To evaluate this integral r must be replaced by its value from the equation of the curve.

Example. Find the area enclosed by the lemniscate

$$r^2 = 2a^2 \cos 2\theta.$$

One loop of the curve (Figure 141) extends from $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$. The area of this loop is

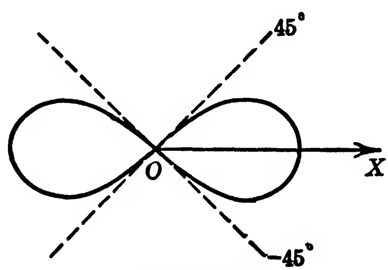

FIGURE 141.

$$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta = \left[\frac{a^2}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = a^2.$$

The entire area is therefore $2a^2$.

PROBLEMS

1. Plot the following points: $A\left(2, \frac{\pi}{6}\right)$, $B\left(2, -\frac{\pi}{6}\right)$, $C\left(-2, \frac{\pi}{6}\right)$, $D\left(-2, -\frac{\pi}{6}\right)$.

2. Plot the following points: $A(3, 120°)$, $B(3, -120°)$, $C(-3, 120°)$, $D(-3, -120°)$.

3. Determine all the pairs of polar coördinates of the point $\left(4, -\frac{\pi}{8}\right)$.

4. Determine all the pairs of polar coördinates of the point $(4, -300°)$.

Sketch the graphs of the following equations:

5. $r = a \sin \theta$.

6. $r = a \sin\left(\theta - \frac{\pi}{4}\right)$.

7. $r = a(1 - \cos \theta)$.

8. $r = a(1 - \sin \theta)$.

9. $r = a \sin 2\theta$.

10. $r = a \cos^2 \theta$.

11. $r = a \sin^3 \theta$.

12. $r = a \sin 3\theta$.

13. $r = a(2 + \sin \theta)$.

14. $r = a(1 + 2 \sin \theta)$.

15. $r = a(1 - 2 \cos \theta)$.

16. $r = a \sin \frac{1}{2}\theta$.

17. $r = a \cos \frac{1}{2}\theta$.

18. $r = a \sin \frac{\theta}{3}$.

19. $r^2 = a^2 \cos \theta$.

20. $r^2 = 2a^2 \sin 2\theta$.

21. $r^2 = a^2(1 - \cos \theta)$.

22. $r = e^\theta$.

23. $r = 4e^{-\frac{\theta}{2}}$.

24. $r = 2 - \theta$.

25. $r\theta = 1$.

26. $r = a \sec \theta$.

27. $r = a \sec \theta + a \csc \theta$.

28. $r = a \tan \theta$.

29. Show that $(1, \frac{3}{4}\pi)$ is on the curve $r = \sin 2\theta$.

30. Show that $(1, \frac{3}{2}\pi)$ is on the curve $r = -2 \sin \frac{\theta}{3}$.

31. If a is constant, show that the equations

$$r = \frac{1}{a \cos \theta + 1}, \quad r = \frac{1}{a \cos \theta - 1}$$

represent the same locus.

32. Show that the equations $r^2 = a^2 \cos^2 2\theta$, $r = a \cos 2\theta$ represent the same locus.

Plot each of the following pairs of curves in one diagram, and find their intersections:

33. $r \sin \theta = a$, $r \cos \theta = a$.

34. $r = 2a \cos \theta$, $r = a$.

35. $r \sin\left(\theta - \frac{\pi}{4}\right) = a$, $r \sin\left(\theta + \frac{\pi}{4}\right) = a$.

36. $r = a \cos\left(\theta - \frac{\pi}{4}\right)$, $r = a \cos\left(\theta + \frac{\pi}{4}\right)$.

37. $r^2 = 2a^2 \cos 2\theta$, $r = a$.

38. $r = a(1 + \cos\theta), r = a(1 + \sin\theta)$.
39. $r = a \sin 2\theta, r = a(1 - \cos 2\theta)$.
40. $r^2 = a^2 \sin\theta, r^2 = a^2 \sin 2\theta$.

By changing to rectangular coördinates determine the loci represented by the following equations:

41. $r \cos\theta = -2$.
42. $r = 4 \sin\theta$.
43. $r = \cos\theta + \sin\theta$.
44. $r = 2 \csc\theta - 3 \sec\theta$.
45. $r^2 - 2r(\cos\theta + \sin\theta) + 1 = 0$.
46. $r(1 - \cos\theta) = 3$.
47. $r(2 - 3\cos\theta) = 6$.
48. $r \cos 2\theta = \sin\theta$.

49. Find the Cartesian equation of the cardioid $r = a(1 + \cos\theta)$.
50. Find the Cartesian equation of the lemniscate $r^2 = 2a^2 \cos 2\theta$.

Determine the polar equations of the following curves:

51. $y^2 = 4x$.
52. $x^2 + y^2 = 2x$.
53. $xy = 1$.
54. $x^2 - y^2 = 1$.
55. $x^2 + y^2 = 4ax + 4ay$.
56. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

57. Find the polar equation of the line through the point $(a, 0)$ perpendicular to the initial line.

58. Find the polar equation of the straight line LK (Figure 132) if N is the point $\left(3, \frac{\pi}{3}\right)$.

59. Find the polar equation of the straight line with intercepts a, b on the x and y axes.

60. Find the polar equation of the circle with center $(a, 0)$ passing through the origin.

61. Find the polar equation of the circle with center $r = a$, $\theta = \frac{\pi}{2}$ passing through the origin.

62. Find the polar equation of the circle through the origin with center $\left(2, \frac{\pi}{6}\right)$.

63. Find the polar equation of the parabola with focus at the origin and vertex $\left(a, \frac{\pi}{2}\right)$.

64. Find the polar equation of the parabola with focus at the origin and vertex $\left(-2, \frac{\pi}{6}\right)$.

65. A variable line through the origin O intersects a fixed line in Q, and the point P is taken on this line such that $OP \cdot OQ = k^2$. Find the locus of P.

66. On the line OQ through the origin O and the point $Q(r, \theta)$ of the curve $r = a \cos\theta$ two points P are taken at distance b from Q. Find the locus of P.

67. A variable line through O intersects a fixed circle in the points Q_1, Q_2, and on this line the point P is taken such that

$$2OP = OQ_1 + OQ_2.$$

Find the locus of P.

68. With the notation of the preceding problem find the locus of P if

$$\frac{2}{OP} = \frac{1}{OQ_1} + \frac{1}{OQ_2}.$$

69. A line of length $2a$ slides with its ends in the x and y axes. Find the locus of the foot of the perpendicular from the origin to the moving line.

70. Find the locus of a point if the product of its distances from two fixed points is equal to the square of half the distance between them.

71. Find the angle ψ on the curve $r = a\theta$.

72. If $0 < \theta < 2\pi$ and $r = a(1 - \cos\theta)$, show that $\psi = \frac{1}{2}\theta$.

73. If $0 < \theta < \frac{\pi}{4}$ and $r^2 = 2a^2 \cos 2\theta$, show that

$$\psi = 2\theta + \tfrac{1}{2}\pi.$$

Find the angles at which the following pairs of curves intersect:

74. $r = 2a\cos\theta,\ r\cos\theta = a.$

75. $r = 2a\cos\theta,\ r = a\sin\theta.$

76. $r = a\cos\theta,\ r = a.$

77. $r = 1 + \sin 2\theta,\ r = 1.$

78. $r = a\sin\frac{1}{2}\theta,\ r = a\sin\theta.$

79. $r^2 = a^2\sin\theta,\ r^2 = a^2(1 - \sin\theta).$

80. Show that the parabolas $r = a\sec^2\frac{\theta}{2}$, $r = b\csc^2\frac{\theta}{2}$ intersect at right angles.

81. Show that the curves $r = ae^{k\theta}$, $r = be^{-\frac{\theta}{k}}$ are perpendicular at each point of intersection.

82. Taking the initial line as x-axis, find the slope of the cardioid

$$r = a(1 + \cos\theta).$$

83. Find the points on the cardioid $r = a(1 - \cos\theta)$ at the greatest distance from the initial line.

84. Let $P(r, \theta)$ be a point on the hyperbola $r^2\sin 2\theta = c$. Show that the triangle formed by the radius OP, the tangent at P, and the initial line is isosceles.

85. Find the circumference of the circle $r = a$ by integration.

86. Find the circumference of the circle $r = 2a\cos\theta$.

87. Find the distance along the line $r = a\sec\left(\theta - \frac{\pi}{3}\right)$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

88. Determine the arc of the logarithmic spiral $r = ae^{k\theta}$ from $\theta = 0$ to the point (r, θ).

89. Show that the arc length of the cardioid $r = a(1 + \cos\theta)$ satisfies the equation

$$ds^2 = 2a^2(1 + \cos\theta)\,d\theta^2 = 4a^2\cos^2\frac{\theta}{2}\,d\theta^2.$$

Determine the perimeter of the cardioid.

90. If $r = a\sin^3\frac{\theta}{3}$, show that

$$ds = a\sin^2\frac{\theta}{3}\,d\theta = \frac{a}{2}(1 - \cos\tfrac{2}{3}\theta)\,d\theta.$$

Determine the complete perimeter of the curve.

91. With suitable conventions as to the angles used, show that the angle from the initial line to the tangent to a curve is

$$\phi = \theta + \psi = \theta + \tan^{-1} \frac{r}{\dfrac{dr}{d\theta}}.$$

By calculating

$$\frac{d\phi}{ds} = \frac{d\phi}{d\theta}\frac{d\theta}{ds}$$

and taking its reciprocal, show that the radius of curvature is

$$\rho = \frac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}.$$

Using the formula in the preceding problem find the radius of curvature of each of the following curves:

92. $r = a\theta.$

93. $r = a(1 + \cos\theta).$

94. $r = ae^{k\theta}.$

95. $r = a\sin^3\dfrac{\theta}{3}.$

96. Find the radius of curvature of the lemniscate

$$r^2 = 2a^2\cos 2\theta$$

at $\theta = \dfrac{\pi}{6}.$

97. Find the area within one loop of the curve

$$r^2 = a^2 \sin \theta.$$

98. Find the area within one loop of the curve

$$r^2 = a^2 \sin n\theta.$$

99. Find the area of the sector bounded by an arc of the curve $r = ae^{k\theta}$ and two radii r_1, r_2.

100. Find the area bounded by an arc of the spiral

$$r = \frac{k}{\theta}$$

and two radii r_1, r_2.

101. By calculating $\frac{1}{2}r^2$ and using the identity

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

determine the area enclosed by the cardioid $r = a(1 + \cos\theta)$.

102. Find the area within the lemniscate

$$r^2 = 2a^2\cos 2\theta$$

and outside the circle $r = a$.

CHAPTER X

VECTORS

104. Scalar and Vector. The simplest quantities of physics, such as mass and temperature, are characterized by magnitude only, or by magnitude and an algebraic sign. Such quantities are called *scalar.* When units of measurement have been chosen a scalar is represented by a real number and graphically by a displacement along a straight line.

In the description of certain quantities direction as well as magnitude is required. Thus to specify the velocity of a moving body it is necessary to state the direction and the speed of motion.

A quantity that is characterized by magnitude and direction is called a *vector.* Force, velocity, and acceleration are illustrations. Such a quantity is represented graphically by an arrow of length equal to its magnitude pointing in the assigned direction.

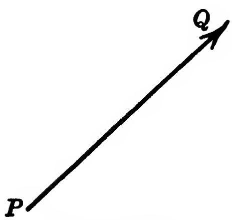

FIGURE 142.

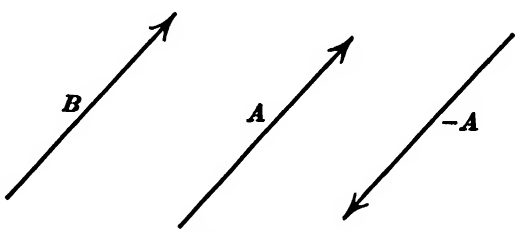

FIGURE 143.

Thus if a point moves from P to Q its displacement is a vector represented by the arrow from P to Q and in symbols by the notation

$$\overrightarrow{PQ}.$$

Two vectors **A, B** are called equal

$$\mathbf{A} = \mathbf{B}$$

if they have the same length and direction. Thus if a rigid body is moved without rotation all its points are moved the same distance

in the same direction. The displacement of each point is a vector, and the displacements of different points are equal vectors.

The symbol $-\mathbf{A}$ is used to represent a vector having the same magnitude as $\mathbf{A}$ but the opposite direction. Thus the displacement from Q to P is the negative of that from P to Q,

$$\overrightarrow{QP} = -\overrightarrow{PQ}.$$

Associated with a vector $\mathbf{A}$ is a positive scalar equal to its length or magnitude. We represent this by the notation $|\mathbf{A}|$. Thus, if a is the length of the vector $\mathbf{A}$,

$$a = |\mathbf{A}|.$$

With respect to multiplication and division we shall find that vectors and scalars satisfy different algebraic laws. It is therefore desirable to have a notation that indicates clearly which quantities in an equation are scalars and which are vectors. In most books vectors are indicated by **bold-faced** type. As this is impossible to indicate in writing, an arrow or a bar is often placed over a letter to show that it represents a vector. Thus $\vec{A}$ or $\bar{A}$ would be understood to represent a vector.

105. Addition and Subtraction of Vectors. Given any two vectors $\mathbf{A}$, $\mathbf{B}$ (Figure 144) we can construct a vector equal to $\mathbf{B}$ with

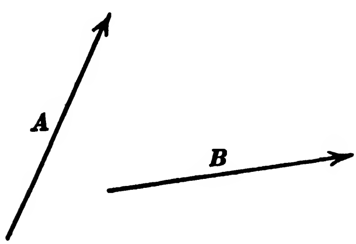

FIGURE 144.

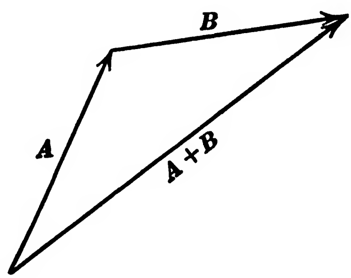

FIGURE 145.

its origin at the end of $\mathbf{A}$ as shown in Figure 145. The vector from the initial point of $\mathbf{A}$ to the terminal point of this vector $\mathbf{B}$ is called the sum of $\mathbf{A}$ and $\mathbf{B}$ and is indicated by the notation $\mathbf{A} + \mathbf{B}$.

To obtain a physical interpretation of this sum consider a point P inside a box. Suppose all points of the box displaced the amount $\mathbf{A}$ and at the same time P displaced with respect to the box the amount $\mathbf{B}$. The total displacement of P in space is then $\mathbf{A} + \mathbf{B}$.

Since the opposite sides of a parallelogram are equal and parallel, it is clear from Figure 146 that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{1}$$

both sums being equal to the vector diagonal **C**. The sum is thus independent of the order in which the vectors are added. This is called the *commutative* law of addition.

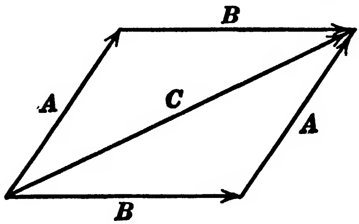

FIGURE 146.

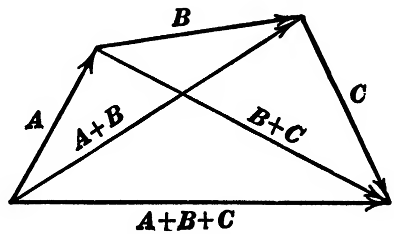

FIGURE 147.

Similarly the sum of three or more vectors is obtained by constructing a polygon (Figure 147) having those vectors as consecutive sides and drawing a vector from the initial point of the first to the terminal point of the last. By constructing the vector diagonals $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{C}$ it is also clear from this diagram that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}). \tag{2}$$

The sum is thus independent of the way its terms are associated into groups. This is called the *associative* law of addition.

If two vectors **A** and **B** are drawn from the same origin (Figure 148) the difference $\mathbf{A} - \mathbf{B}$ is defined as the vector extending

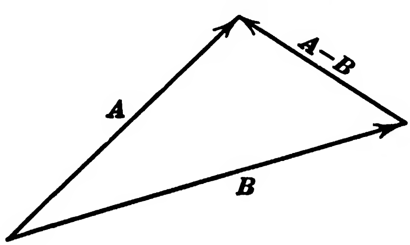

FIGURE 148.

from the end of **B** to the end of **A**. This diagram also shows that

$$\mathbf{B} + (\mathbf{A} - \mathbf{B}) = \mathbf{A},$$

which states that $\mathbf{A} - \mathbf{B}$ is the vector which added to **B** gives **A**.

106. Multiplication by a Scalar. If m is a scalar and $\mathbf{A}$ is a vector, the notation

$$m\mathbf{A} = \mathbf{A}m$$

is used to represent a vector m times as long as $\mathbf{A}$ and having the same direction if m is positive, the opposite direction if m is negative. If m, n are scalars, it follows that

$$(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A} \tag{1}$$

since both sides represent a vector $m + n$ times as long as $\mathbf{A}$, having the same direction if $m + n$ is positive, the opposite direction if $m + n$ is negative.

Similarly, if m is a scalar and $\mathbf{A}$, $\mathbf{B}$ are vectors,

$$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}. \tag{2}$$

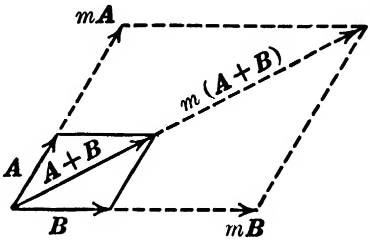

FIGURE 149.

For this equation merely states that, if the sides $\mathbf{A}$, $\mathbf{B}$, and the diagonal $\mathbf{A} + \mathbf{B}$ of a parallelogram (Figure 149) are all increased in the same proportion, the new vectors $m\mathbf{A}$, $m\mathbf{B}$, $m(\mathbf{A} + \mathbf{B})$ are the sides and the diagonal of a similar parallelogram.

Equations (1) and (2) express what is called the *distributive* law for the multiplication of scalars and vectors.

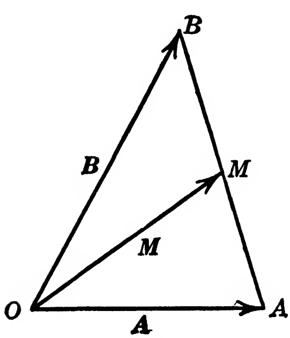

FIGURE 150.

Example 1. If $\mathbf{A}$, $\mathbf{B}$ are vectors from the point O to the points $\mathbf{A}$, $\mathbf{B}$, find the vector from O to the middle point of AB.

Let $\mathbf{M}$ be the vector from O to the middle point M. The vector from A to M is $\mathbf{M} - \mathbf{A}$, and that from M to B is $\mathbf{B} - \mathbf{M}$. Since these are equal we have

$$\mathbf{M} - \mathbf{A} = \mathbf{B} - \mathbf{M},$$

whence

$$\mathbf{M} = \tfrac{1}{2}(\mathbf{A} + \mathbf{B}).$$

Example 2. $\overrightarrow{AB} = \overrightarrow{CD} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{DB}$, and $\overrightarrow{DF} = \frac{1}{3}\overrightarrow{DA}$. Find $\overrightarrow{CE}$ and $\overrightarrow{CF}$, and show that C, F, E are on a line.

The required arrangement of points is shown in Figure 151. We have

$$\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DE} = \mathbf{u} - \frac{\mathbf{v}}{2},$$

$$\overrightarrow{CF} = \overrightarrow{CD} + \overrightarrow{DF} = \mathbf{u} - \tfrac{1}{3}(\mathbf{u} + \mathbf{v}) = \tfrac{2}{3}\mathbf{u} - \tfrac{1}{3}\mathbf{v}.$$

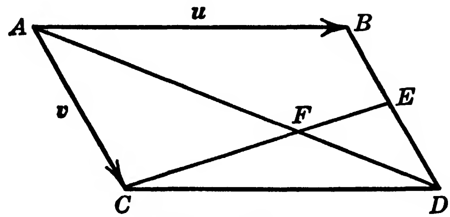

FIGURE 151.

Comparison of these expressions shows that

$$\overrightarrow{CF} = \tfrac{2}{3}\overrightarrow{CE}.$$

The vectors $\overrightarrow{CE}$ and $\overrightarrow{CF}$ thus extend from C in the same direction. The points C, E, F are therefore on a line.

Example 3. The vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ form consecutive sides of a regular octagon. Express $\mathbf{A}_3$ in terms of $\mathbf{A}_1$ and $\mathbf{A}_2$.

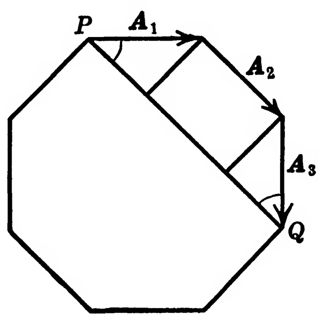

FIGURE 152.

In Figure 152, $\overrightarrow{PQ}$ is parallel to $\mathbf{A}_2$ and the angles at P and Q are 45°. Since $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are of equal length

$$PQ = |\mathbf{A}_1| \cos 45° + |\mathbf{A}_2| + |\mathbf{A}_3| \cos 45° = (1 + \sqrt{2})\,|\mathbf{A}_2|.$$

Thus

$$\overrightarrow{PQ} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = (1 + \sqrt{2})\mathbf{A}_2,$$

and so

$$\mathbf{A}_3 = \sqrt{2}\mathbf{A}_2 - \mathbf{A}_1.$$

107. Points on a Line. Let **A, B, P** be the vectors from a point O to the points A, B, P. The vector from A to P is $\mathbf{P} - \mathbf{A}$, and that from A to B is $\mathbf{B} - \mathbf{A}$. If P lies on the line AB and divides it in the ratio

$$\frac{AP}{AB} = m, \tag{1}$$

these vectors have the same ratio, that is,

$$\mathbf{P} - \mathbf{A} = m(\mathbf{B} - \mathbf{A}),$$

or

$$\mathbf{P} = m\mathbf{B} + (1 - m)\mathbf{A}. \tag{2}$$

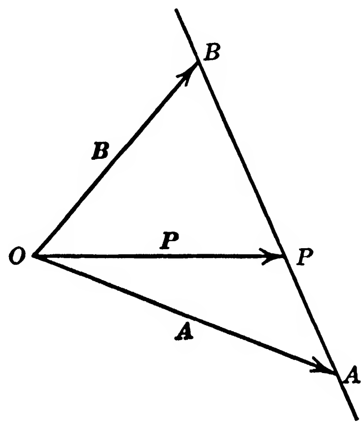

FIGURE 153.

The vector **P** is thus a linear function of **A** and **B** with coefficients whose sum is

$$m + 1 - m = 1.$$

Conversely, if

$$\mathbf{P} = m\mathbf{A} + n\mathbf{B} \tag{3}$$

and

$$m + n = 1, \tag{4}$$

the point P lies on the line AB and

$$\frac{AP}{PB} = \frac{n}{m}.$$

For (3), in virtue of (4), can be written

$$(m + n)\mathbf{P} = m\mathbf{A} + n\mathbf{B},$$

whence

$$m(\mathbf{P} - \mathbf{A}) = n(\mathbf{B} - \mathbf{P}).$$

Example 1. Find the point P on the line AB three-fourths of the way from A to B.

The vector from O to P may be obtained from equation (2) with $m = \frac{3}{4}$ or from (3) with $m = \frac{1}{4}$, $n = \frac{3}{4}$, the result in either case being

$$\mathbf{P} = \tfrac{1}{4}\mathbf{A} + \tfrac{3}{4}\mathbf{B}.$$

Example 2. $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are vectors from O to the points A, B, C, D, and

$$\overrightarrow{AB} = 2\overrightarrow{CD}. \tag{5}$$

Determine the vector from O to the point P in which BC and AD intersect.

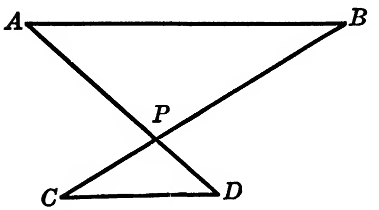

FIGURE 154.

Equation (5) is equivalent to

$$\mathbf{B} - \mathbf{A} = 2(\mathbf{D} - \mathbf{C}).$$

Transposing $\mathbf{B}$ and $\mathbf{C}$ to one side, $\mathbf{A}$ and $\mathbf{D}$ to the other, this becomes

$$\mathbf{B} + 2\mathbf{C} = \mathbf{A} + 2\mathbf{D}$$

whence

$$\frac{\mathbf{B} + 2\mathbf{C}}{3} = \frac{\mathbf{A} + 2\mathbf{D}}{3}.$$

The left side of this equation represents the vector from O to a point on BC; the right side, the vector from O to a point on AD. Thus either of these represents the vector from O to P.

108. Center of Gravity, Centroid. If masses $m_1, m_2, \cdots, m_n$ are located at the points $P_1, P_2, \cdots, P_n$, the center of gravity of these masses may be defined as the point C such that

$$m_1\overrightarrow{CP_1} + m_2\overrightarrow{CP_2} + \cdots + m_n\overrightarrow{CP_n} = 0. \tag{1}$$

Let $\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_n$ be the vectors from a point O to the points $P_1, P_2, \cdots, P_n$, and $\mathbf{C}$ the vector from O to the center of gravity. Equation (1) is then equivalent to

$$m_1(\mathbf{P}_1 - \mathbf{C}) + m_2(\mathbf{P}_2 - \mathbf{C}) + \cdots + m_n(\mathbf{P}_n - \mathbf{C}) = 0,$$

whence

$$\mathbf{C} = \frac{m_1\mathbf{P}_1 + m_2\mathbf{P}_2 + \cdots + m_n\mathbf{P}_n}{m_1 + m_2 + \cdots + m_n} \tag{2}$$

is the vector from O to the center of gravity.

It should be noted that the vectors in (1) depend only on the points $C, P_1, P_2, \cdots, P_n$. Thus the point determined by (2) is independent of the point O from which the vectors $\mathbf{C}, \mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_n$ are drawn.

If all the coefficients $m_1, m_2, \cdots, m_n$ are equal, the point C determined by (2) is called the *centroid* of the n points. Thus, if $\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_n$ are the vectors from O to the points $P_1, P_2, \cdots, P_n$,

$$\mathbf{C} = \frac{\mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n}{n} \tag{3}$$

is the vector from O to the centroid of those points. The centroid of a set of points is often called their center of gravity, even when no weights or masses are associated with the points.

109. Components. By the angle θ between two vectors $\mathbf{V}_1$ $\mathbf{V}_2$ (Figure 155) we mean the angle having those vectors as terminal sides with arrows pointing away from the vertex. In many cases

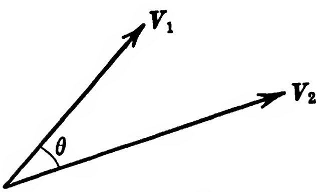

FIGURE 155.

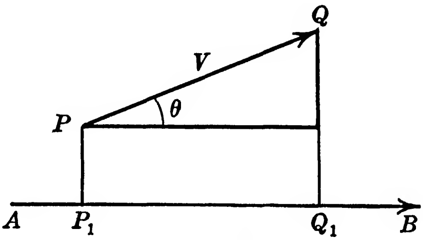

FIGURE 156.

only the cosine of the angle is used, and then its direction is immaterial. But, if direction is significant, the angle must be considered as extending from one vector to the other with algebraic sign defined as in §56.

A vector $\mathbf{V}$ (Figure 156) extends from an initial point P to a terminal point Q. Its projection on a directed line $\overrightarrow{AB}$ extends from the projection of P to the projection of Q. This projection P_1Q_1, considered as a scalar with algebraic sign positive or negative according as it extends in the direction $\overrightarrow{AB}$ or its opposite, is called the *component* of $\mathbf{V}$ along $\overrightarrow{AB}$. If θ is the angle between $\mathbf{V}$ and $\overrightarrow{AB}$ this component is evidently

$$|\mathbf{V}| \cos \theta.$$

When rectangular axes are used it is customary to represent vectors of unit length and positive direction along OX and OY by the letters $\mathbf{i}$ and $\mathbf{j}$. Any vector $\mathbf{V} = \overrightarrow{PQ}$ in the plane is the sum of two vectors

$$\mathbf{V} = \overrightarrow{PR} + \overrightarrow{RQ}$$

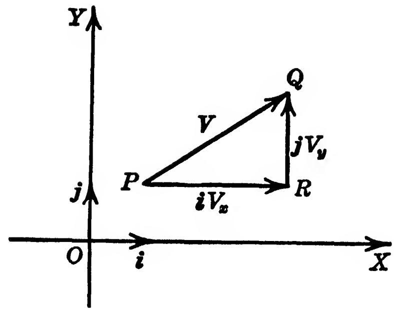

FIGURE 157.

parallel to OX and OY (Figure 157). If V_x and V_y are the components of **V** along the axes, these vectors have the form

$$\overrightarrow{PR} = \mathbf{i}V_x, \quad \overrightarrow{RQ} = \mathbf{j}V_y$$

and so

$$\mathbf{V} = \mathbf{i}V_x + \mathbf{j}V_y. \tag{1}$$

Being the hypotenuse of a right triangle with sides V_x and V_y, the length of **V** is

$$|\mathbf{V}| = \sqrt{V_x^2 + V_y^2}. \tag{2}$$

In particular, the vector $\mathbf{r} = \overrightarrow{OP}$ from the origin to the point $P(x, y)$ has the components x, y (Figure 158). This vector

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y \tag{3}$$

is often used to represent the variable point $P(x, y)$.

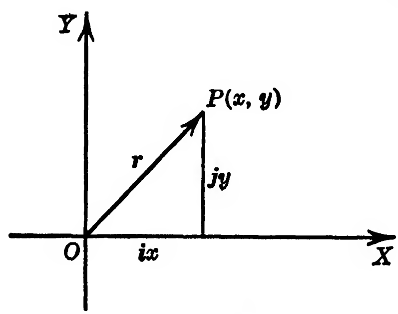

FIGURE 158.

If **U** and **V** are two vectors with components U_x, U_y and V_x, V_y

$$\mathbf{U} = \mathbf{i}U_x + \mathbf{j}U_y, \quad \mathbf{V} = \mathbf{i}V_x + \mathbf{j}V_y$$

and

$$\mathbf{U} + \mathbf{V} = \mathbf{i}(U_x + V_x) + \mathbf{j}(U_y + V_y).$$

The components of the sum are thus obtained by adding corresponding components.

Example 1. Determine the vector from $P(-2, 3)$ to $Q(4, -2)$.

The components are evidently the differences of the coördinates (end minus beginning) at the two ends of the vector (Figure 159).

Thus

$$\overrightarrow{PQ} = 6\mathbf{i} - 5\mathbf{j}.$$

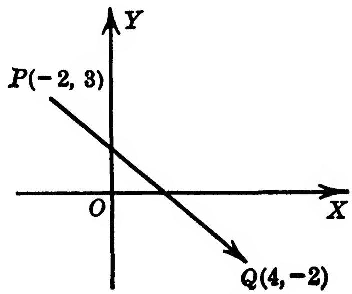

FIGURE 159.

Example 2. Find the vector from the origin to the centroid of the triangle formed by the points $P_1(1, -2)$, $P_2(3, 4)$, $P_3(5, 1)$.

The vectors from the origin to the three points are

$$\mathbf{P}_1 = \mathbf{i} - 2\mathbf{j},$$
$$\mathbf{P}_2 = 3\mathbf{i} + 4\mathbf{j},$$
$$\mathbf{P}_3 = 5\mathbf{i} + \mathbf{j}.$$

The vector from the origin to the centroid is thus

$$\mathbf{C} = \tfrac{1}{3}(\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3) = 3\mathbf{i} + \mathbf{j}.$$

The centroid is the point (3, 1).

110. Vector Function of a Scalar Variable, Derivative. If to each value of a scalar variable t there corresponds a value of a vector $\mathbf{V}$, then $\mathbf{V}$ is called a function of t, a relation indicated by the notation

$$\mathbf{V} = \mathbf{F}(t).$$

Thus, if t is the time and $\mathbf{r} = \overrightarrow{OP}$ is the vector from a fixed point O to a variable point P, then $\mathbf{r}$ is a function of t.

When t approaches t_0 the vector $\mathbf{F}(t)$ is said to approach the constant vector $\mathbf{A}$ as limit if the magnitude of the difference $\mathbf{F}(t) - \mathbf{A}$ tends to the limit zero. That is,

$$\lim_{t \to t_0} \mathbf{F}(t) = \mathbf{A}$$

and

$$\lim_{t \to t_0} |\mathbf{F}(t) - \mathbf{A}| = 0$$

are by definition equivalent equations. If $\mathbf{A}$ is not zero, this means that when t is sufficiently near t_0 the vector $\mathbf{F}(t)$ has almost

the direction as well as almost the magnitude of $\mathbf{A}$. But if $\mathbf{A} = 0$ the direction of $\mathbf{F}(t)$ may vary arbitrarily provided merely that $|\mathbf{F}(t)|$ tends to zero.

When t changes to $t + \Delta t$ the change in $\mathbf{F}(t)$ is

$$\Delta\mathbf{F} = \mathbf{F}(t + \Delta t) - \mathbf{F}(t).$$

Since Δt is scalar the ratio

$$\frac{\Delta\mathbf{F}}{\Delta t}$$

is a vector parallel to $\Delta\mathbf{F}$. If this ratio tends to a limit when Δt tends to zero, that limit, written

$$\frac{d\mathbf{F}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{F}}{\Delta t}$$

is called the derivative of $\mathbf{F}$ with respect to t. In particular, if t is the time, $\frac{d\mathbf{F}}{dt}$ is called the *rate of change* of $\mathbf{F}$.

In proving the formulas of differentiation no direct use was made of the fact that the expression differentiated was scalar, but merely that it satisfied certain algebraic laws (such as the distributive law of multiplication) and that it had certain continuity properties. Formulas that involve only addition and multiplication by a scalar are therefore also valid for vector functions. Thus, if u and $\mathbf{V}$ are differentiable, u a scalar and $\mathbf{V}$ a vector,

$$\frac{d}{dt}(u\mathbf{V}) = u\frac{d\mathbf{V}}{dt} + \mathbf{V}\frac{du}{dt}.$$

To prove this one would calculate $\Delta(u\mathbf{V})$, divide by Δt, and take the limit just as if $\mathbf{V}$ were scalar.

Example. If $\mathbf{A}$ and $\mathbf{B}$ are vector constants, determine the first and second derivatives of

$$\mathbf{V} = \mathbf{A}\cos\theta + \mathbf{B}\sin\theta$$

with respect to θ.

Differentiating as if $\mathbf{A}$ and $\mathbf{B}$ were scalar constants, we get

$$\frac{d\mathbf{V}}{d\theta} = -\mathbf{A}\sin\theta + \mathbf{B}\cos\theta,$$

$$\frac{d^2\mathbf{V}}{d\theta^2} = -\mathbf{A}\cos\theta - \mathbf{B}\sin\theta.$$

111. Velocity and Acceleration. If a particle P moves along a straight or curved path, its vector displacement

$$\mathbf{r} = \overrightarrow{OP}$$

from a fixed point O is a function of the time t. The rate of change of displacement

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{1}$$

is called the *velocity*, and the magnitude of velocity $|\mathbf{v}|$ is called the *speed* of P.

To determine these quantities let P be the position of the particle at time t, Q its position at time $t + \Delta t$, and s the arc along the path from a fixed point P_0 to P. During the interval Δt the changes in $\mathbf{r}$ and s are $\Delta\mathbf{r} = \overrightarrow{PQ}$ and $\Delta s = \text{arc } PQ$. Also

$$\frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta\mathbf{r}}{\Delta s}\frac{\Delta s}{\Delta t}. \tag{2}$$

The vector

$$\frac{\Delta\mathbf{r}}{\Delta s} = \frac{\overrightarrow{PQ}}{\Delta s}$$

extends along the line PQ and is of length

$$\frac{|\Delta\mathbf{r}|}{|\Delta s|} = \frac{\text{chord } PQ}{\text{arc } PQ}.$$

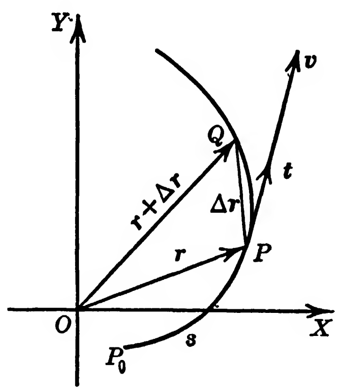

FIGURE 160.

As Δs tends to zero, Q approaches P, the line PQ approaches the tangent, and the ratio of chord to arc approaches unity. Thus $\dfrac{\Delta\mathbf{r}}{\Delta s}$ approaches a vector of unit length

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} \tag{3}$$

tangent to the curve at P.

In virtue of (1) and (3), equation (2) gives

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{t}\frac{ds}{dt} \tag{4}$$

as limit. The velocity is therefore tangent to the curve at P, and the speed

$$|\mathbf{v}| = \left|\frac{ds}{dt}\right|$$

is equal to the rate of change of distance along the curve.

The rate of change of velocity

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{5}$$

is called the *acceleration* of the particle.

If O is taken as origin and x, y are the rectangular coördinates of P,

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y.$$

Since $\mathbf{i}$ and $\mathbf{j}$ are constants,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt}, \tag{6}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{i}\frac{d^2x}{dt^2} + \mathbf{j}\frac{d^2y}{dt^2}. \tag{7}$$

Thus the velocity and acceleration of a moving point are vectors with components along OX and OY equal to the first and second derivatives of x and y with respect to t.

The speed of the moving particle is

$$|\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{8}$$

Example 1. At time t the coördinates of a moving point are

$$x = t\cos t, \quad y = t\sin t.$$

Find its components of velocity and acceleration along the coördinate axes at time $t = 0$.

The components of velocity are

$$\frac{dx}{dt} = \cos t - t\sin t = 1,$$

$$\frac{dy}{dt} = \sin t + t\cos t = 0.$$

The components of acceleration are

$$\frac{d^2x}{dt^2} = -2 \sin t - t \cos t = 0,$$

$$\frac{d^2y}{dx^2} = 2 \cos t - t \sin t = 2.$$

Example 2. At time t the coördinates of a moving point P are

$$x = a \cos \omega t, \quad y = a \sin \omega t,$$

a and ω being constants. Find its velocity, acceleration, and speed.

The displacement of P from the origin is

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y = \mathbf{i}a \cos \omega t + \mathbf{j}a \sin \omega t.$$

The velocity and acceleration are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mathbf{i}a\omega \sin \omega t + \mathbf{j}a\omega \cos \omega t,$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\mathbf{i}a\omega^2 \cos \omega t - \mathbf{j}a\omega^2 \sin \omega t.$$

Comparison of the expressions for $\mathbf{a}$ and $\mathbf{r}$ shows that

$$\mathbf{a} = -\omega^2\mathbf{r}.$$

The acceleration is thus directed toward the origin and in magnitude equals ω^2 times the distance. The speed is

$$|\,\mathbf{v}\,| = \sqrt{a^2\omega^2 \sin^2 \omega t + a^2\omega^2 \cos^2 \omega t} = a\omega.$$

112. Tangential and Normal Components. If

$$\mathbf{r} = \overrightarrow{OP}$$

is the vector from a fixed point O to a variable point P on a curve, and s is arc length measured along the curve from a fixed point P_0 to P, we have found that

$$\frac{d\mathbf{r}}{ds} = \mathbf{t} \tag{1}$$

is a unit vector tangent to the curve at P [§111, (3)]. Along the curve (Figure 161) this vector may be considered a function of s.

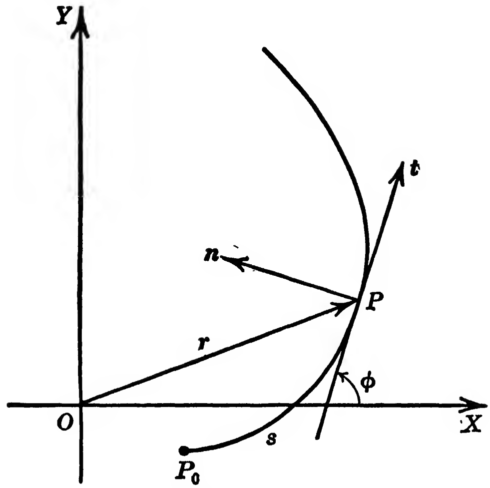

FIGURE 161.

To determine its derivative construct a vector $\overrightarrow{CA}$ equal to $\mathbf{t}$ with its origin at a fixed point C (Figure 162). As P moves along its curve the end A of this vector describes a circle of unit radius. If CB is taken parallel to the x-axis, the arc BA on this circle is equal to the angle ϕ from the x-axis to the tangent at P. By equation (1)

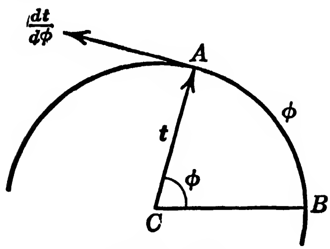

FIGURE 162.

$$\frac{d\mathbf{t}}{d\phi}$$

is then a unit vector $\mathbf{n}$ tangent to the circle and so perpendicular to $\mathbf{t}$. Also

$$\frac{d\phi}{ds} = \frac{1}{\rho}$$

is the curvature at P (§96). Thus

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{d\phi}\frac{d\phi}{ds} = \frac{\mathbf{n}}{\rho}$$

extends along the normal at P and has a magnitude equal to the curvature. We can therefore write

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{t}}{ds} = \frac{\mathbf{n}}{\rho} \tag{2}$$

where $\mathbf{n}$ is the unit normal at P and ρ is the radius of curvature.

From a diagram it is furthermore clear that this vector is directed toward the concave side of the curve. If the radius of curvature is considered positive (as is usual), the unit vector **n** thus extends along the normal toward the concave side of the curve.

Suppose that a particle moves along the curve and at time t has the position P. Its velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{t}\frac{ds}{dt}. \tag{3}$$

Differentiating with respect to t and using the equation

$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds}\frac{ds}{dt} = \frac{\mathbf{n}}{\rho}\frac{ds}{dt}$$

we obtain for the acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{t}\frac{d^2s}{dt^2} + \frac{\mathbf{n}}{\rho}\left(\frac{ds}{dt}\right)^2. \tag{4}$$

The vectors **t** and **n** are unit vectors along perpendicular directions. The coefficients of **t** and **n** in (4) are therefore the components of acceleration along those directions. Calling these components a_t and a_n, we thus have

$$\text{tangential component} = a_t = \frac{d^2s}{dt^2}, \tag{5}$$

$$\text{normal component} = a_n = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2. \tag{6}$$

When the absolute value

$$a = |\mathbf{a}|$$

of the acceleration is known the equation

$$a^2 = a_n^2 + a_t^2 = a_n^2 + \left(\frac{d^2s}{dt^2}\right)^2$$

gives

$$a_n = \sqrt{a^2 - \left(\frac{d^2s}{dt^2}\right)^2} \tag{7}$$

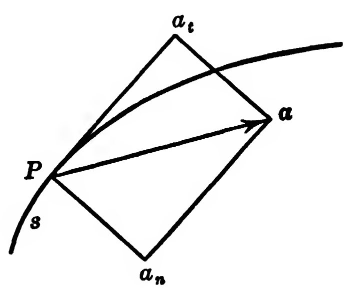

FIGURE 163.

as the normal component of acceleration. Since

$$a^2 = a_x^2 + a_y^2 = \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2$$

this can also be written

$$a_n = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 - \left(\frac{d^2s}{dt^2}\right)^2}. \tag{8}$$

Example. A particle P moves in a parabola, its position at time t being

$$x = bt, \quad y = ct - \tfrac{1}{2}gt^2.$$

Find its tangential and normal components of acceleration.

The vector from the origin to P is

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y = \mathbf{i}bt + \mathbf{j}(ct - \tfrac{1}{2}gt^2).$$

The velocity, speed, and acceleration are therefore

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt), \tag{9}$$

$$|\,\mathbf{v}\,| = \frac{ds}{dt} = \sqrt{b^2 + (c - gt)^2}, \tag{10}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\mathbf{j}g. \tag{11}$$

By equations (5) and (10) the tangential component of acceleration is

$$a_t = \frac{d^2s}{dt^2} = -\frac{g(c - gt)}{\sqrt{b^2 + (c - gt)^2}}.$$

Since the absolute value of the acceleration is

$$|\,\mathbf{a}\,| = |\,-\mathbf{j}g\,| = g,$$

equation (7) gives for the normal component

$$a_n = \sqrt{g^2 - \left(\frac{d^2s}{dt^2}\right)^2} = \frac{bg}{\sqrt{b^2 + (c - gt)^2}}.$$

113. Polar Coördinates. When the position of a moving point P is represented by polar coördinates, components of velocity and acceleration are usually taken along the radius vector $\mathbf{r} = \overrightarrow{OP}$ and perpendicular to it. Let $\mathbf{u}_r$ be the unit vector along $\mathbf{r}$ in the direction of increasing r and $\mathbf{u}_\theta$ the unit vector perpendicular to $\mathbf{r}$ in the direction of increasing θ.

Along a curve (Figure 164) $\mathbf{u}_r$ and $\mathbf{u}_\theta$ may be considered functions of θ. To determine the derivatives of these functions construct vectors equal to $\mathbf{u}_r$ and $\mathbf{u}_\theta$ extending from a fixed center C (Figure 165). The ends of these vectors lie on a unit circle, and

θ is the arc from a radius CB parallel to OX to the end of $\mathbf{u}_r$. Thus

$$\frac{d\mathbf{u}_r}{d\theta}$$

is a unit vector tangent to the circle, and it is evident from the

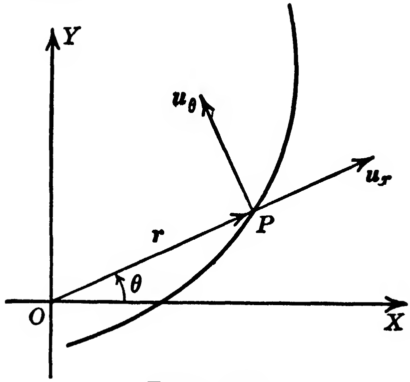

FIGURE 164.

diagram that this vector has the same direction as $\mathbf{u}_\theta$. Similarly,

$$\frac{d\mathbf{u}_\theta}{d\theta}$$

is a unit vector with direction opposite to $\mathbf{u}_r$. Therefore

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta, \quad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r. \tag{1}$$

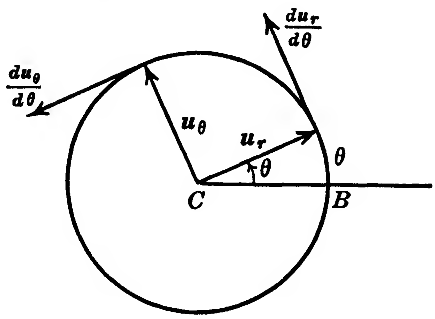

FIGURE 165.

If a particle moves along the curve and at time t has the position $P(r, \theta)$,

$$\mathbf{r} = r\mathbf{u}_r. \tag{2}$$

By differentiating with respect to the time, using equations (1) and relations of the form

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt},$$

we obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{u}_r \frac{dr}{dt} + \mathbf{u}_\theta r \frac{d\theta}{dt}, \tag{3}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{u}_r \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] + \mathbf{u}_\theta \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right] \tag{4}$$

as the velocity and acceleration of the particle.

Denoting components along $\mathbf{r}$ and perpendicular to $\mathbf{r}$ by the subscripts r and θ, we thus have

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt}, \tag{5}$$

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \quad a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}, \tag{6}$$

as the components of velocity and acceleration. It should be noted that these are not merely the first and second derivatives as in rectangular coördinates.

Example. A particle moves in the parabola

$$r = \frac{2p}{1 - \cos\theta} \tag{7}$$

with constant speed v. Find its acceleration.

Writing

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2},$$

we have

$$r = p\csc^2\frac{\theta}{2},$$

$$\frac{dr}{dt} = -p\csc^2\frac{\theta}{2}\cot\frac{\theta}{2}\frac{d\theta}{dt},$$

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = p^2\csc^6\frac{\theta}{2}\left(\frac{d\theta}{dt}\right)^2.$$

If we assume that θ is between 0° and 180° and is increasing, the last two equations give

$$\frac{d\theta}{dt} = \frac{v}{p}\sin^3\frac{\theta}{2}, \tag{8}$$

$$\frac{dr}{dt} = -v\cos\frac{\theta}{2}. \tag{9}$$

Since the tangent line makes equal angles $\frac{1}{2}\theta$ with the radius vector and a line parallel to the axis (Problem 89, page 142) these equations merely state that $\dfrac{dr}{dt}$ and $r\dfrac{d\theta}{dt}$ are the components of velocity along the radius vector and perpendicular to it. By differentiating a second time and substituting in (6), we obtain

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{v^2}{2p}\sin^4\frac{\theta}{2}, \tag{10}$$

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = -\frac{v^2}{2p}\sin^3\frac{\theta}{2}\cos\frac{\theta}{2}. \tag{11}$$

By use of (8) these components can be written

$$a_r = -\frac{v}{2}\frac{d\theta}{dt}\sin\frac{\theta}{2},$$

$$a_\theta = -\frac{v}{2}\frac{d\theta}{dt}\cos\frac{\theta}{2}.$$

These equations show that the acceleration makes an angle $\dfrac{\pi}{2} - \dfrac{\theta}{2}$ with the line PO (Figure 166) and is therefore normal to the curve, as it must

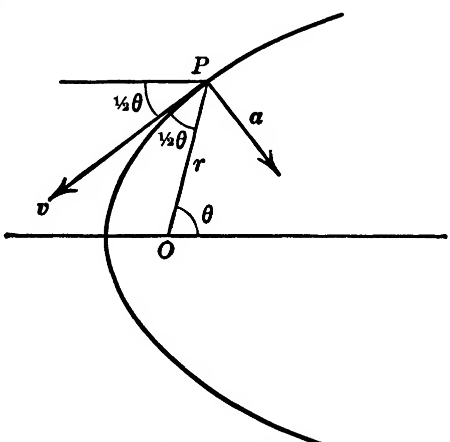

FIGURE 166.

be since the speed is constant. The magnitude of the acceleration is

$$\frac{v}{2}\frac{d\theta}{dt}.$$

This also follows directly from the fact that the speed is constant and the vector $\mathbf{v}$ makes the angle $\frac{1}{2}\theta$ with a fixed direction. For, if we draw a vector

$$\overrightarrow{CA} = \mathbf{v}$$

from a fixed point C, the acceleration

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\overrightarrow{CA}$$

is the velocity of the point A in a rotation with angular velocity $\dfrac{d}{dt}(\frac{1}{2}\theta)$ around a circle of radius v.

PROBLEMS

1. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are vectors from a point O to the points A, B, C, D, and

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D},$$

show that $ABCD$ is a parallelogram.

2. $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$ are sides of a square $ABCD$, and E is the middle point of CD. Express $\overrightarrow{AE}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

3. If $\overrightarrow{AD} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BE} = \frac{2}{3}\overrightarrow{BC}$, express $\overrightarrow{DE}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

4. If $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$, and D and E are the middle points of AB and BC, express $\overrightarrow{AE}$, $\overrightarrow{CD}$, and $\overrightarrow{DE}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

5. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors from the origin to the points A, B, C, and D is the point on the line BC one-third of the way from B to C, find $\overrightarrow{AD}$.

6. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors from a point O to the points A, B, C, and D is the point on the line AB one-third of the way from A to B. If E is the middle point of AC, find $\overrightarrow{DE}$.

7. From the vertices of a triangle vectors are drawn to the middle points of the opposite sides. Show that the sum of these vectors is zero.

8. If $\overrightarrow{AB} = \overrightarrow{CD} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, $\overrightarrow{AP} = \frac{1}{4}\mathbf{u}$, and $\overrightarrow{CQ} = \frac{3}{4}\mathbf{u}$, express $\overrightarrow{PQ}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

9. If $\overrightarrow{AB} = \overrightarrow{CD} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, $\overrightarrow{DE} = \frac{1}{3}\overrightarrow{DB}$, and $\overrightarrow{DF} = \frac{1}{4}\overrightarrow{DA}$, express $\overrightarrow{CE}$ and $\overrightarrow{CF}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, and show that the points C, F, E are on a line.

10. $\mathbf{A}$, $\mathbf{B}$ are vectors forming consecutive sides of a regular hexagon. Find the vectors which form the other four sides.

11. The vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ form consecutive sides of a regular polygon of n sides. Express $\mathbf{A}_3$ in terms of $\mathbf{A}_1$ and $\mathbf{A}_2$.

12. A, B, C, D are four points so placed that

$$\overrightarrow{AB} = \overrightarrow{DC},$$

M is the middle point of BC, and P is the point on BD one-third of the way from B to D. Show that A, P, M are on a line.

13. If A, B, C, D are four points so placed that

$$\overrightarrow{AB} = \overrightarrow{CD},$$

E is on the line BC two-thirds of the way from B to C, and $\overrightarrow{BD} = \overrightarrow{DF}$, show that A, E, F are on a line.

14. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors extending from a point. If three numbers x, y, z (not all zero) can be found such that

$$x\mathbf{A} + y\mathbf{B} + z\mathbf{C} = 0,$$

$$x + y + z = 0,$$

show that the ends of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are on a line.

15. Three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ extend from points on a line. Determine the condition that their ends lie on a parallel line.

16. If $\overrightarrow{OP_1} = \mathbf{A}$, $\overrightarrow{OP_2} = \mathbf{B}$, and

$$\overrightarrow{OP_3} = \mathbf{A}\sin^2 t + \mathbf{B}\cos^2 t,$$

show that P_1, P_2, P_3 are on a line.

17. If $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ are vectors from O to points P_1, P_2, P_3 on a line and m_1, m_2, m_3 are any numbers with sum equal to 1, show that

$$\mathbf{P} = m_1\mathbf{P}_1 + m_2\mathbf{P}_2 + m_3\mathbf{P}_3$$

is the vector from O to a point P on the same line.

18. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are vectors from O to the points A, B, C, D, and

$$\overrightarrow{AB} = m\overrightarrow{CD}.$$

Find the vector from O to the point P in which BC and AD intersect.

19. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors from O to the points A, B, C, and D, E are points on the lines AB, BC such that

$$\frac{AD}{DB} = \frac{n}{m}, \quad \frac{BE}{EC} = \frac{p}{n}.$$

Determine the vector from O to the point P in which the lines AE and CD intersect.

20. M is the middle point of AB and P the point on the line MC one-third of the way from M to C. Show that P is the centroid of A, B, C.

21. M_1 is the middle point of AB, M_2 the middle point of CD, and M the middle point of M_1M_2. Show that M is the centroid of A, B, C, D.

22. If the masses $m_1, m_2, \cdots, m_n$ are located at points $P_1, P_2, \cdots, P_n$ on a line, show that their center of gravity is on the same line.

23. The length of a vector $\mathbf{V}$ is 4, and the angle from the x-axis to $\mathbf{V}$ is 60°. Find the components of $\mathbf{V}$ along OX and OY.

24. Given $A(2, -3)$, $B(-1, 4)$, $C(3, 5)$, express $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ in terms of $\mathbf{i}$ and $\mathbf{j}$, and show that the sum of these vectors is zero.

25. Show that the vectors

$$\mathbf{V}_1 = 2\mathbf{i} - 4\mathbf{j},$$

$$\mathbf{V}_2 = \mathbf{i} + 2\mathbf{j},$$

$$\mathbf{V}_3 = 2\mathbf{j} - 3\mathbf{i}$$

form the sides of a triangle.

26. Find the slope of the line along which the vector

$$\mathbf{V} = \mathbf{i}a + \mathbf{j}b$$

extends.

27. Show that the vectors

$$\mathbf{V}_1 = \mathbf{i} + 3\mathbf{j},\ \mathbf{V}_2 = 2\mathbf{i} + 6\mathbf{j}$$

are parallel.

28. Show that the vectors

$$\mathbf{V}_1 = \mathbf{i}a + \mathbf{j}b,\ \mathbf{V}_2 = \mathbf{j}a - \mathbf{i}b$$

are perpendicular.

29. Find the components of $3\mathbf{i}$ and $4\mathbf{j}$ along the vector

$$\mathbf{V} = 3\mathbf{i} + 4\mathbf{j}.$$

30. Find the component of the vector $5\mathbf{i} + \mathbf{j}$ along the vector $5\mathbf{i} - 12\mathbf{j}$.

31. Show that the vectors $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_n$ have a sum equal to zero if and only if the sum of their components along every direction is zero.

32. The vector $3\mathbf{i} - 2\mathbf{j}$ extends from $P(-4, 5)$ to $Q(x, y)$. Find x and y.

33. Given $A(1, 3)$, $B(5, -1)$, $C(3, 2)$, find the coördinates of D if

$$\overrightarrow{AB} = 2\overrightarrow{CD}.$$

34. Weights of 1, 3, and 5 pounds are placed at the points $P_1(2, -3)$, $P_2(4, 3)$, $P_3(-1, 6)$ respectively. Find the center of gravity of these weights.

35. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{i}'$, $\mathbf{j}'$ be unit vectors along two sets of rectangular axes OX, OY and OX', OY'. If x, y and x', y' are coördinates of the same point P with respect to these axes, show that

$$\mathbf{i}x + \mathbf{j}y = \mathbf{i}'x' + \mathbf{j}'y'.$$

36. If $\mathbf{A}$ and $\mathbf{B}$ are vector constants and

$$\mathbf{F}(t) = \mathbf{A}\sin 2t + \mathbf{B}(1 - \cos 2t),$$

find $\mathbf{F}(0)$ and

$$\lim_{t \to 0} \frac{\mathbf{F}(t)}{t}.$$

37. The point $P(x, y)$ moves along the curve

$$y = 2x^2 - x.$$

If $\mathbf{r} = \overrightarrow{OP}$ is the vector from the origin to P, find

$$\lim_{x \to 0} \left(\frac{\mathbf{r}}{x}\right).$$

38. $\overrightarrow{OP}$ is the vector from the origin to the point $P(r, \theta)$ on the curve $r\theta = 1$. Find

$$\lim_{r \to 0} (\overrightarrow{OP}).$$

39. If θ is the angle from the x-axis to a given direction, show that

$$\mathbf{u} = \mathbf{i} \cos \theta + \mathbf{j} \sin \theta$$

is a unit vector along that direction. Find the first and second derivatives of $\mathbf{u}$ with respect to θ, and interpret the results graphically.

40. Determine the vector constants $\mathbf{C}_1$, $\mathbf{C}_2$ such that

$$\mathbf{V} = \mathbf{C}_1 \cos \theta + \mathbf{C}_2 \sin \theta$$

satisfies the equation

$$\frac{d^2\mathbf{V}}{d\theta^2} - \mathbf{V} = \mathbf{A} \sin \theta.$$

In the following problems $\mathbf{r} = \overrightarrow{OP}$ is the vector from the origin to a moving point P. Find its velocity, acceleration, and speed at the time indicated.

41. $\mathbf{r} = \mathbf{i}t^2 - 3\mathbf{j}t, \quad t = 2.$
42. $\mathbf{r} = \mathbf{i}(t^3 - 1) + \mathbf{j}(t^2 - 1), \quad t = 1.$
43. $\mathbf{r} = 3\mathbf{i} \sin t + 2\mathbf{j} \cos t, \quad t = \dfrac{\pi}{2}.$
44. $\mathbf{r} = 3\mathbf{i}e^t + 4\mathbf{j}e^{-t}, \quad t = 0.$
45. $\mathbf{r} = \mathbf{i} \ln \sin t + \mathbf{j}t, \quad t = \dfrac{\pi}{6}.$
46. $\mathbf{r} = \mathbf{i} \tan t + \mathbf{j} \cot t, \quad t = \dfrac{\pi}{4}.$

In the following problems find the components of velocity and acceleration, and the speed, of the point $P(x, y)$ at time t:

47. $x = 24t$, $y = 8t - 16t^2$.
48. $x = t + \frac{1}{2}t^2$, $y = t - \frac{1}{2}t^2$.
49. $x = t \cos t - \sin t$, $y = t \sin t + \cos t$.
50. $x = \ln \cos t$, $y = t$.

51. At time t the coördinates of a moving point are

$$x = a_1t^2 + b_1t + c_1,$$

$$y = a_2t^2 + b_2t + c_2,$$

a_1, b_1, c_1, a_2, b_2, c_2 being constants. Show that the acceleration is constant.

52. At time t the coördinates of a moving point are

$$x = \frac{a}{2}(e^{kt} + e^{-kt}), \quad y = \frac{a}{2}(e^{kt} - e^{-kt}).$$

Show that its acceleration is proportional to the distance from the origin and directed away from it.

53. A point $P(x, y)$ moves in the parabola $y^2 = 4px$ in such a way that $\dfrac{dy}{dt}$ is constant. Show that its acceleration is constant.

54. A point $P(x, y)$ moves in the hyperbola $xy = c$ in such a way that $\dfrac{dx}{dt} = u$ is constant. Find its acceleration.

In each of the following problems find the tangential and normal components of acceleration:

55. $x = 2\cos t,\ y = \sin t.$
56. $x = at^2,\ y = bt^2.$
57. $x = a\cos kt,\ y = a\sin kt.$
58. $x = \dfrac{a}{2}(e^t + e^{-t}),\ y = \dfrac{a}{2}(e^t - e^{-t}).$
59. $x = a(\phi - \sin\phi),\ y = a(1 - \cos\phi), \quad \phi = kt.$
60. $x = a(\phi\cos\phi - \sin\phi),\ y = a(\phi\sin\phi + \cos\phi), \quad \phi = kt.$

61. If a particle moves in a curve with speed v and acceleration a, show that the radius of curvature ρ of the path satisfies the equation

$$\frac{1}{\rho} = \frac{\sqrt{a^2 - \left(\dfrac{dv}{dt}\right)^2}}{v^2}.$$

In the following problems find the velocity, acceleration, and speed of the point $P(r, \theta)$ at time t:

62. $r = a\theta,\ \theta = \omega t.$
63. $r = a(t + c)^2,\ \theta = k\ln(t + c).$
64. $r = a(t + c),\ \theta = \dfrac{k}{t + c}.$
65. $r = \dfrac{a}{2}(e^\theta - e^{-\theta}),\ \theta = \omega t.$
66. $r = ae^{-kt},\ \theta = be^{2kt}.$

67. The point P moves with constant speed v along the radius vector OP while OP turns about O with constant angular velocity ω. Find the acceleration of P.

68. A projectile P moves in a parabola under the constant acceleration g of gravity. If r is the distance from the focus of the parabola to P, show that

$$\frac{d^2r}{dt^2} = g.$$

69. A particle moves with constant speed v in the hyperbola $r^2\sin 2\theta = c$. Find its acceleration. Show that it is perpendicular to the path and of magnitude (cf. Problem 84, page 216)

$$v\frac{d\theta}{dt}.$$

70. A particle moves with constant speed v in the cardioid

$$r = a(1 - \cos\theta).$$

Find its acceleration. Show that it is perpendicular to the path and of magnitude (cf. Problem 72, page 216)

$$\frac{3}{2}v\frac{d\theta}{dt}.$$

CHAPTER XI

FORMULAS AND METHODS OF INTEGRATION

114. Elementary Functions. The algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions and those expressible by finite combinations of these functions are sometimes called *elementary* Thus

$$\log\left(\frac{\sin x}{e^x + \sqrt{x^2 - 1}}\right)$$

is an elementary function.

By means of the differentiation formulas (§§21, 78, 80, 87) the derivative of any elementary function can be obtained, the result being again an elementary function. The indefinite integral of an elementary function may not, however, be elementary. Thus

$$\int e^{-x^2}\,dx$$

is not expressible as an elementary function of x.

In the integrations we shall perform the result will be an elementary function. Since no such function may exist, the method of finding it cannot be entirely systematic. The general procedure is to try to express the given differential as a sum of parts each of which can be integrated by a known formula. A list of such formulas is given in the following section. More extensive lists are found in *integral tables.*

115. Formulas. In the following formulas u is any variable or function of a single variable, du is its differential, n is constant, and C is the constant of integration. In evaluating a definite integral C may be omitted, but in all other cases it should be added.

I. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$, if n is not -1.

II. $\displaystyle\int \frac{du}{u} = \ln|u| + C.$

III. $\displaystyle\int \sin u\, du = -\cos u + C.$

IV. $\displaystyle\int \cos u\, du = \sin u + C.$

V. $\displaystyle\int \sec^2 u\, du = \tan u + C.$

VI. $\displaystyle\int \csc^2 u\, du = -\cot u + C.$

VII. $\displaystyle\int \sec u \tan u\, du = \sec u + C.$

VIII. $\displaystyle\int \csc u \cot u\, du = -\csc u + C.$

IX. $\displaystyle\int \tan u\, du = -\ln |\cos u| + C.$

X. $\displaystyle\int \cot u\, du = \ln |\sin u| + C.$

XI. $\displaystyle\int \sec u\, du = \ln |\sec u + \tan u| + C.$

XII. $\displaystyle\int \csc u\, du = \ln |\csc u - \cot u| + C.$

XIII. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C.$

XIV. $\displaystyle\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$

XV. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C.$

XVI. $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln |u + \sqrt{u^2 \pm a^2}| + C.$

XVII. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C.$

XVIII. $\displaystyle\int e^u\, du = e^u + C.$

It should be noted that logarithms only of absolute values occur in these formulas. In the applications the absolute value signs are usually omitted, it being assumed that the work has been so arranged that logarithms of negative numbers do not occur. Also it is understood that all square roots are positive and inverse trigonometric functions are represented by principal values (§79).

Any one of these formulas can be proved by showing that the differential of the right member is equal to the expression under the integral sign. Thus to prove XI we note that

$$|\sec u + \tan u| = \pm(\sec u + \tan u),$$

the sign on the right depending on the value of u. Hence

$$d \ln |\sec u + \tan u| = \frac{\pm(\sec u \tan u + \sec^2 u)\, du}{\pm(\sec u + \tan u)} = \sec u\, du.$$

116. Integration by Substitution. When some function of the variable is taken as u, a given integral may assume the form of one of the integration formulas or differ from such form only by a constant factor. Integration accomplished in this way is called integration by substitution.

In each integration formula the expression integrated is of the form $f(u)\,du$. To reduce a given differential to this form it is necessary to choose u such that one factor becomes $f(u)$ and the remainder becomes du. Failure to make this second factor exactly du is responsible for most errors in integration.

Example 1. $\int x\sqrt{2x^2 - 1}\, dx.$

The differential of $2x^2 - 1$ differs only by a constant factor from $x\, dx$. Thus we take

$$2x^2 - 1 = u, \quad 4x\, dx = du.$$

Then

$$\int x\sqrt{2x^2 - 1}\, dx = \int \tfrac{1}{4}u^{\frac{1}{2}}\, du = \tfrac{1}{6}u^{\frac{3}{2}} + C = \tfrac{1}{6}(2x^2 - 1)^{\frac{3}{2}} + C.$$

Example 2. $\int \frac{\cos 2x\, dx}{1 + \sin 2x}.$

We observe that $\cos 2x\, dx$ differs only by a constant factor from the differential of $1 + \sin 2x$. Thus we let

$$1 + \sin 2x = u.$$

Then

$$2 \cos 2x\, dx = du, \quad \cos 2x\, dx = \tfrac{1}{2}\, du,$$

and

$$\int \frac{\cos 2x\, dx}{1 + \sin 2x} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln (1 + \sin 2x) + C.$$

Example 3. $\int \frac{dt}{t\sqrt{4t^2 - 9}}$.

This suggests formula XV. To bring it to exactly that form let

$$u = 2t, \quad a = 3.$$

Then

$$\begin{aligned}\int \frac{dt}{t\sqrt{4t^2 - 9}} &= \int \frac{2dt}{2t\sqrt{4t^2 - 9}} \\ &= \int \frac{du}{u\sqrt{u^2 - a^2}} \\ &= \frac{1}{a}\sec^{-1}\frac{u}{a} + C \\ &= \frac{1}{3}\sec^{-1}\frac{2t}{3} + C.\end{aligned}$$

Example 4. $\int e^{\tan x}\sec^2 x\, dx$.

If $u = \tan x$, by formula XVIII

$$\int e^{\tan x}\sec^2 x\, dx = \int e^u\, du = e^{\tan x} + C.$$

117. Powers of Trigonometric Functions. A power of a trigonometric function multiplied by its differential can be integrated by formula I. Thus, if $u = \tan x$,

$$\int \tan^3 x \cdot \sec^2 x\, dx = \int u^3\, du = \tfrac{1}{4}\tan^4 x + C.$$

Differentials can often be reduced to this form by trigonometric transformations. The general procedure is to take as u a trigonometric function such that one factor of the given differential is du and the remaining factors can be expressed in terms of u *without introducing radicals.* This is illustrated by the following examples.

Example 1. $\int \sin^2 x \cos^3 x\, dx$.

If we take $\cos x\, dx$ as $d(\sin x)$ and use the relation $\cos^2 x = 1 - \sin^2 x$, the other factors can be expressed in terms of $\sin x$ without introducing radicals. Thus

$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= \int \sin^2 x \cdot \cos^2 x \cdot \cos x\, dx \\ &= \int \sin^2 x(1 - \sin^2 x)\, d(\sin x) \\ &= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C.\end{aligned}$$

Example 2. $\int \tan x \sec^4 x \, dx.$

If we take $\sec^2 x$ as $d(\tan x)$ and use the relation $\sec^2 x = 1 + \tan^2 x$, the other factors can be expressed in terms of $\tan x$ without introducing radicals. Thus

$$\begin{aligned}\int \tan x \sec^4 x \, dx &= \int \tan x \cdot \sec^2 x \cdot \sec^2 x \, dx \\ &= \int \tan x (1 + \tan^2 x) \, d(\tan x) \\ &= \tfrac{1}{2} \tan^2 x + \tfrac{1}{4} \tan^4 x + C.\end{aligned}$$

Example 3. $\int \tan^3 x \sec x \, dx.$

If we take $\tan x \sec x \, dx$ as $d(\sec x)$ and use the relation

$$\tan^2 x = \sec^2 x - 1,$$

the integral takes the form

$$\begin{aligned}\int \tan^3 x \sec x \, dx &= \int \tan^2 x \cdot \tan x \sec x \, dx \\ &= \int (\sec^2 x - 1) \, d(\sec x) \\ &= \tfrac{1}{3} \sec^3 x - \sec x + C.\end{aligned}$$

Example 4. $\int \tan^5 x \, dx.$

If we replace $\tan^2 x$ by $\sec^2 x - 1$, the integral becomes

$$\int \tan^5 x \, dx = \int \tan^3 x (\sec^2 x - 1) \, dx = \tfrac{1}{4} \tan^4 x - \int \tan^3 x \, dx.$$

The integral is thus made to depend on a simpler one

$$\int \tan^3 x \, dx.$$

Similarly

$$\int \tan^3 x \, dx = \int \tan x (\sec^2 x - 1) \, dx = \tfrac{1}{2} \tan^2 x + \ln |\cos x|.$$

Hence finally

$$\int \tan^5 x \, dx = \tfrac{1}{4} \tan^4 x - \tfrac{1}{2} \tan^2 x - \ln |\cos x| + C.$$

118. Products of Sines and Cosines. Integrals of the form

$$\int \sin^m x \cos^n x \, dx \tag{1}$$

can be evaluated by the methods of the preceding section if at least one of the exponents m or n is an odd positive integer. If both

m and n are even, however, those methods fail. In that event we can make the integral depend on others of simpler form by means of the formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}, \tag{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}, \tag{3}$$

$$\sin u \cos u = \frac{\sin 2u}{2}. \tag{4}$$

Products of sines and cosines of different angles can be expressed as sums or differences by means of the formulas

$$\sin A \sin B = \tfrac{1}{2}[\cos (A - B) - \cos (A + B)], \tag{5}$$

$$\sin A \cos B = \tfrac{1}{2}[\sin (A - B) + \sin (A + B)], \tag{6}$$

$$\cos A \cos B = \tfrac{1}{2}[\cos (A - B) + \cos (A + B)]. \tag{7}$$

Example 1. $\int \cos^4 x \sin^2 x \, dx.$

By formulas (2), (3), (4) we have

$$\begin{aligned}\cos^4 x \sin^2 x &= \cos^2 x \cdot \cos^2 x \sin^2 x \\ &= \frac{1 + \cos 2x}{2} \frac{\sin^2 2x}{4} \\ &= \tfrac{1}{8}[\sin^2 2x + \sin^2 2x \cos 2x] \\ &= \frac{1}{8}\left[\frac{1 - \cos 4x}{2} + \sin^2 2x \cos 2x\right].\end{aligned}$$

Consequently

$$\int \cos^4 x \sin^2 x \, dx = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

Example 2. $\int \sin 2x \cos 3x \, dx.$

By formula (6)

$$\sin 2x \cos 3x = \tfrac{1}{2}[\sin (-x) + \sin 5x] = \tfrac{1}{2}[\sin 5x - \sin x].$$

Hence

$$\int \sin 2x \cos 3x \, dx = -\tfrac{1}{10} \cos 5x + \tfrac{1}{2} \cos x + C.$$

119. Quadratic Expressions. Formulas XIII–XVII contain quadratic expressions in which only the second power of the variable appears. Integrals containing the general quadratic

$ax^2 + bx + c$ can be reduced to this form by completing the square of $ax^2 + bx$. Before doing this, however, it may be helpful to separate a part in which $ax^2 + bx + c$ can be used as new variable.

Example 1. $\displaystyle\int \frac{dx}{3x^2 + 6x + 15}$.

Completing the square of the quadratic in the denominator, we get

$$3x^2 + 6x + 15 = 3(x^2 + 2x + 1) + 12 = 3(x + 1)^2 + 12.$$

Taking $u = x + 1$ we thus have

$$\begin{aligned}\int \frac{dx}{3x^2 + 6x + 15} &= \int \frac{du}{3u^2 + 12} \\ &= \frac{1}{3}\int \frac{du}{u^2 + 4} \\ &= \frac{1}{6}\tan^{-1}\frac{u}{2} + C \\ &= \frac{1}{6}\tan^{-1}\frac{x + 1}{2} + C.\end{aligned}$$

Example 2. $\displaystyle\int \frac{dx}{\sqrt{2 + 2x - 3x^2}}$.

The coefficient of x^2 being negative, we place the terms containing x in parentheses preceded by -3. Thus

$$2 + 2x - 3x^2 = 2 - 3(x^2 - \tfrac{2}{3}x) = \tfrac{7}{3} - 3(x - \tfrac{1}{3})^2.$$

If $u = x - \frac{1}{3}$, we then have

$$\begin{aligned}\int \frac{dx}{\sqrt{2 + 2x - 3x^2}} &= \int \frac{du}{\sqrt{\frac{7}{3} - 3u^2}} \\ &= \frac{1}{\sqrt{3}}\int \frac{du}{\sqrt{\frac{7}{9} - u^2}} \\ &= \frac{1}{\sqrt{3}}\sin^{-1}\frac{u}{\sqrt{\frac{7}{9}}} + C \\ &= \frac{1}{\sqrt{3}}\sin^{-1}\frac{3x - 1}{\sqrt{7}} + C.\end{aligned}$$

Example 3. $\displaystyle\int \frac{(2x - 1)\,dx}{\sqrt{4x^2 + 4x + 2}}$.

Since the numerator contains the first power of x, we resolve the integral into two parts

$$\int \frac{(2x - 1)\,dx}{\sqrt{4x^2 + 4x + 2}} = \frac{1}{4}\int \frac{(8x + 4)\,dx}{\sqrt{4x^2 + 4x + 2}} - 2\int \frac{dx}{\sqrt{4x^2 + 4x + 2}}$$

In the first integral on the right the numerator is taken equal to the differential of $4x^2 + 4x + 2$. In the second the numerator is dx. The outside factors $\frac{1}{4}$ and -2 are chosen so that the two sides of the equation are equal. The first integral has the form

$$\frac{1}{4}\int \frac{du}{\sqrt{u}} = \tfrac{1}{2}\sqrt{u} = \tfrac{1}{2}\sqrt{4x^2 + 4x + 2}.$$

The second integral is evaluated by completing the square. The final result is

$$\int \frac{(2x - 1)\,dx}{\sqrt{4x^2 + 4x + 2}} =$$

$$\tfrac{1}{2}\sqrt{4x^2 + 4x + 2} - \ln\left(2x + 1 + \sqrt{4x^2 + 4x + 2}\,\right) + C.$$

120. Rational Fractions. A fraction, such as

$$\frac{x^3 + 3x}{x^2 - 2x - 3},$$

whose numerator and denominator are polynomials, is called a *rational fraction.*

If the degree of the numerator is equal to or greater than that of the denominator, the fraction should be reduced by division. Thus

$$\frac{x^3 + 3x}{x^2 - 2x - 3} = x + 2 + \frac{10x + 6}{x^2 - 2x - 3}.$$

A fraction with numerator of lower degree than its denominator can be resolved into a sum of *partial fractions* with denominators that are factors of the original denominator. Thus

$$\frac{10x + 6}{x^2 - 2x - 3} = \frac{10x + 6}{(x - 3)(x + 1)} = \frac{9}{x - 3} + \frac{1}{x + 1}.$$

These fractions can often be found by trial. If not, proceed as in the following examples. The procedure is one in which the order of steps can be inverted. That the given expression is resolvable into fractions of the kind suggested is in each case thus shown by finding the fractions.

CASE 1. *Factors of the denominator all of the first degree and none repeated.*

Example 1. $\displaystyle\int \frac{x^4 + 2x + 6}{x^3 + x^2 - 2x}\,dx.$

Dividing numerator by denominator, we get

$$\frac{x^4 + 2x + 6}{x^3 + x^2 - 2x} = x - 1 + \frac{3x^2 + 6}{x^3 + x^2 - 2x} = x - 1 + \frac{3x^2 + 6}{x(x - 1)(x + 2)}.$$

Assume that

$$\frac{3x^2 + 6}{x(x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 2}, \tag{1}$$

A, B, C being constants. The two sides of this equation are merely different ways of writing the same function. If then we clear of fractions, the two sides of the resulting equation

$$\begin{aligned} 3x^2 + 6 &= A(x - 1)(x + 2) + Bx(x + 2) + Cx(x - 1) \\ &= (A + B + C)x^2 + (A + 2B - C)x - 2A, \end{aligned} \tag{2}$$

are identical. That is,

$$A + B + C = 3, \quad A + 2B - C = 0, \quad -2A = 6. \tag{3}$$

Solving these equations, we get

$$A = -3, \quad B = 3, \quad C = 3.$$

Conversely, if A, B, C have these values, equations (3), and so (2) and (1), are satisfied. Therefore

$$\begin{aligned} \int \frac{x^4 + 2x - 6}{x^3 + x^2 - 2x}\,dx &= \int \left(x - 1 - \frac{3}{x} + \frac{3}{x - 1} + \frac{3}{x + 2}\right) dx \\ &= \tfrac{1}{2}x^2 - x - 3 \ln x + 3 \ln (x - 1) \\ &\qquad + 3 \ln (x + 2) + C \\ &= \tfrac{1}{2}x^2 - x + 3 \ln \frac{(x - 1)(x + 2)}{x} + C. \end{aligned}$$

The constants can often be determined more easily by substituting particular values for x on the two sides of the equation after clearing fractions. Thus the equation (2) above,

$$3x^2 + 6 = A(x - 1)(x + 2) + Bx(x + 2) + Cx(x - 1),$$

is an identity satisfied for all values of x. In particular, if $x = 0$, it becomes

$$6 = -2A,$$

whence $A = -3$. Similarly, by substituting $x = 1$ and $x = -2$ we get

$$9 = 3B, \quad 18 = 6C$$

whence $B = 3$, $C = 3$.

CASE 2. *Factors of the denominator all of first degree but some repeated.*

Example 2. $\displaystyle\int \frac{8x^3 + 7}{(x + 1)(2x + 1)^3}\,dx.$

Assume that

$$\frac{8x^3 + 7}{(x + 1)(2x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(2x + 1)^3} + \frac{C}{(2x + 1)^2} + \frac{D}{2x + 1}. \tag{5}$$

Corresponding to the repeated factor $(2x + 1)^3$ we introduce fractions with $(2x + 1)^3$ and all lower powers as denominators. Clearing and solving as before, we find

$$A = 1, \quad B = 12, \quad C = -6, \quad D = 0.$$

Hence

$$\int \frac{8x^3 + 7}{(x + 1)(2x + 1)^3}\,dx = \int \left[\frac{1}{x + 1} + \frac{12}{(2x + 1)^3} - \frac{6}{(2x + 1)^2}\right] dx$$
$$= \ln (x + 1) - \frac{3}{(2x + 1)^2} + \frac{3}{2x + 1} + C.$$

CASE 3. *Denominator containing factors of the second degree but none repeated.*

Example 3. $\displaystyle\int \frac{4x^2 + x + 1}{x^3 - 1}\,dx.$

The denominator can be factored in the form

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

Assume

$$\frac{4x^2 + x + 1}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}.$$

With the quadratic denominator $x^2 + x + 1$ we use a numerator that is not a single constant but a linear function $Bx + C$. Clearing fractions and solving for A, B, C, we find

$$A = 2, \quad B = 2, \quad C = 1.$$

Therefore

$$\int \frac{4x^2 + x + 1}{x^3 - 1}\,dx = \int \left(\frac{2}{x - 1} + \frac{2x + 1}{x^2 + x + 1}\right) dx$$
$$= 2 \ln (x - 1) + \ln (x^2 + x + 1) + C.$$

CASE 4. *Denominator containing factors of the second degree, some being repeated.*

Example 4. $\displaystyle\int \frac{1}{x(x^2 + 1)^2}\,dx.$

Assume

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{(x^2 + 1)^2} + \frac{Dx + E}{x^2 + 1}.$$

Corresponding to the repeated second degree factor $(x^2 + 1)^2$, we introduce partial fractions having as denominators $(x^2 + 1)^2$ and all lower powers of $x^2 + 1$, all the numerators being of first degree. Clearing fractions and solving for A, B, C, D, we find

$$A = 1, \quad B = -1, \quad C = 0, \quad D = -1, \quad E = 0.$$

Hence

$$\int \frac{1}{x(x^2+1)^2}\,dx = \int \left[\frac{1}{x} - \frac{x}{(x^2+1)^2} - \frac{x}{x^2+1}\right] dx$$
$$= \ln \frac{x}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + C.$$

121. Change of Variable. An integral

$$\int f(x)\,dx \tag{1}$$

may not be in form for direct evaluation by the methods we have described but may be reducible to such form by a change of variable. When x is replaced by

$$x = \phi(t) \tag{2}$$

and dx by

$$dx = \phi'(t)\,dt \tag{3}$$

the above integral becomes

$$\int f(\phi(t))\,\phi'(t)\,dt.$$

Suppose that this can be integrated and the result is

$$\int f(\phi(t))\,\phi'(t)\,dt = \Phi(t) + C. \tag{4}$$

This is the indefinite integral expressed as a function of t. To express it in terms of x we solve equation (2) for t in terms of x and substitute in the right side of (4). If in this way we obtain

$$\Phi(t) = F(x), \tag{5}$$

then

$$\int f(x)\,dx = F(x) + C. \tag{6}$$

In the case of a definite integral it is usually simpler not to express the final result in terms of x but to change the limits.

Suppose that t_1, t_2 are the values of t that correspond to the values x_1, x_2 of x. From (5) we have

$$\Phi(t_1) = F(x_1), \quad \Phi(t_2) = F(x_2).$$

Hence

$$F(x_2) - F(x_1) = \Phi(t_2) - \Phi(t_1).$$

From (6) and (4) this is equivalent to

$$\int_{x_1}^{x_2} f(x)\, dx = \int_{t_1}^{t_2} f(\phi(t))\, \phi'(t)\, dt. \tag{7}$$

To change the variable in a definite integral we thus merely replace x and dx by their values in terms of the new variable and replace the limits by the corresponding values of the new variable. In the above discussion it has been assumed that, when t varies from t_1 to t_2, x varies from x_1 to x_2, and that for all intermediate values t is a one-valued function of x, and conversely. If these conditions are not satisfied equation (7) may not be valid.

122. Integrals Containing $(ax + b)^{\frac{p}{q}}$. If p and q are positive integers, the substitution

$$ax + b = z^q$$

can be used to express x, dx, and

$$(ax + b)^{\frac{p}{q}}$$

as rational functions of z and dz. If several fractional powers of the same linear function $ax + b$ occur, the substitution

$$(ax + b) = z^n$$

may be used, n being so chosen that all the roots can be extracted.

Example 1. $\displaystyle\int \frac{dx}{1 + \sqrt{x - 1}}$.

Let $x - 1 = z^2$. Then $dx = 2z\, dz$ and

$$\begin{aligned}
\int \frac{dx}{1 + \sqrt{x-1}} &= \int \frac{2z\, dz}{1 + z} \\
&= \int \left(2 - \frac{2}{1 + z}\right) dz \\
&= 2z - 2\ln(1 + z) + C \\
&= 2\sqrt{x - 1} - 2\ln(1 + \sqrt{x - 1}) + C.
\end{aligned}$$

Example 2. $\displaystyle\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{5}{6}}}$.

Let $x = z^6$. Then $dx = 6z^5\,dz$ and

$$\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{5}{6}}} = \int \frac{6z^5\,dz}{z^3 + z^5}$$
$$= \int 6\left(1 - \frac{1}{1 + z^2}\right) dz$$
$$= 6z - 6\tan^{-1} z + C$$
$$= 6x^{\frac{1}{6}} - 6\tan^{-1}(x^{\frac{1}{6}}) + C.$$

123. Trigonometric Substitutions. If a differential contains $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, one should first consider using the square root as a new variable. If that substitution fails, the differential can often be integrated by taking x, a, and the square root as the three sides of a right triangle and using one of its acute angles as new variable.

Example 1. $\displaystyle\int_0^a x(a^2 - x^2)^{\frac{3}{2}}\,dx$.

Let $(a^2 - x^2)^{\frac{1}{2}} = z$. We then have

$$a^2 - x^2 = z^2, \quad -2x\,dx = 2z\,dz, \quad x\,dx = -z\,dz.$$

When x varies from 0 to a, z varies from a to 0. Thus

$$\int_0^a x(a^2 - x^2)^{\frac{3}{2}}\,dx = \int_0^a (a^2 - x^2)^{\frac{3}{2}}\,x\,dx$$
$$= \int_a^0 z^3(-z\,dz)$$
$$= \int_0^a z^4\,dz$$
$$= \frac{a^5}{5}.$$

Example 2. $\displaystyle\int \sqrt{a^2 - x^2}\,dx$.

Take a as hypotenuse and x as one side of a right triangle (Figure 167).

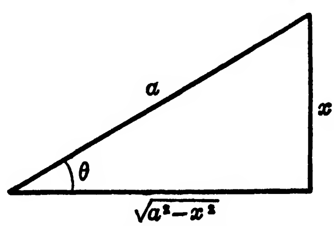

FIGURE 167.

Then

$$x = a \sin \theta, \quad dx = a \cos \theta \, d\theta, \quad \sqrt{a^2 - x^2} = a \cos \theta,$$

and

$$\int \sqrt{a^2 - x^2}\, dx = \int a^2 \cos^2 \theta \, d\theta$$

$$= \int \frac{a^2}{2} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2} [\theta + \tfrac{1}{2} \sin 2\theta] + C$$

$$= \frac{a^2}{2} [\theta + \sin \theta \cos \theta] + C.$$

From the triangle

$$\theta = \sin^{-1} \frac{x}{a}, \quad \sin \theta = \frac{x}{a}, \quad \cos \theta = \frac{\sqrt{a^2 - x^2}}{a}.$$

Hence

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C.$$

Example 3. $\displaystyle\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}.$

Take x as hypotenuse and a as base of a right triangle (Figure 168)

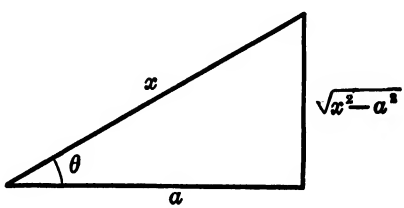

FIGURE 168.

Then

$$x = a \sec \theta, \quad dx = a \sec \theta \tan \theta \, d\theta, \quad \sqrt{x^2 - a^2} = a \tan \theta,$$

and

$$\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}} = \int \frac{a \sec \theta \tan \theta \, d\theta}{a^3 \tan^3 \theta}$$

$$= \int \frac{1}{a^2} \csc \theta \cot \theta \, d\theta$$

$$= -\frac{1}{a^2} \csc \theta + C$$

$$= -\frac{x}{a^2 \sqrt{x^2 - a^2}} + C.$$

Example 4. $\int_0^a \frac{dx}{(x^2 + a^2)^2}$.

Take a and x as the two sides of a right triangle (Figure 169). Then

$$x = a \tan \theta, \quad dx = a \sec^2 \theta \, d\theta, \quad a^2 + x^2 = a^2 \sec^2 \theta.$$

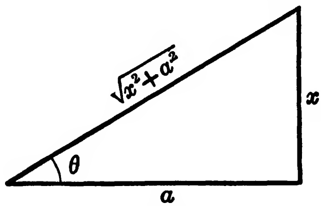

FIGURE 169.

When x varies from 0 to a, θ varies from 0 to $\frac{\pi}{4}$. Consequently,

$$\int_0^a \frac{dx}{(x^2 + a^2)^2} = \frac{1}{a^3} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sec^2 \theta}$$

$$= \frac{1}{a^3} \int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta$$

$$= \frac{1}{2a^3} [\theta + \sin \theta \cos \theta]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2a^3} \left[\frac{\pi}{4} + \frac{1}{2}\right].$$

124. Integration by Parts. From the formula

$$d(uv) = u\,dv + v\,du$$

we get

$$u\,dv = d(uv) - v\,du,$$

whence

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

If $\int v\,du$ is known, this gives $\int u\,dv$. Integration by the use of this formula is called *integration by parts.*

Example 1. $\int \ln x \, dx$.

Let $u = \ln x$, $dv = dx$. Then

$$du = \frac{dx}{x}, \quad v = x$$

and

$$\int \ln x \, dx = (\ln x)\, x - \int x \cdot \frac{dx}{x}$$

$$= x\,(\ln x - 1) + C.$$

Example 2. $\int x^2 \sin x\, dx.$

Let $u = x^2$, $dv = \sin x\, dx$. Then $du = 2x\, dx$, $v = -\cos x$, and

$$\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx.$$

A second integration by parts with $u = 2x$, $dv = \cos x\, dx$ gives

$$\int 2x \cos x\, dx = 2x \sin x - \int 2 \sin x\, dx = 2x \sin x + 2 \cos x + C.$$

Hence finally

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

The method of integration by parts applies particularly well to functions that are simplified by differentiation, like $\ln x$, or to products of functions of different classes, like $x \sin x$. In applying the method the given differential must be resolved into a product $u \cdot dv$. The part called dv must have a known integral, and the part called u should usually be simplified by differentiation.

Sometimes after integration by parts a multiple of the original differential appears on the right side of the equation. It can be transposed to the other side and the integral solved for algebraically. This is illustrated by the following examples.

Example 3. $\int \sqrt{a^2 - x^2}\, dx.$

Integrating by parts with $u = \sqrt{a^2 - x^2}$, $dv = dx$, we get

$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} - \int - \frac{x^2\, dx}{\sqrt{a^2 - x^2}}.$$

Adding a^2 to the numerator of the integral and subtracting an equivalent integral, this becomes

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} \\ &= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}.\end{aligned}$$

Transposing the second term on the right and dividing by 2, we get

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C.$$

Example 4. $\int e^{ax} \cos bx\, dx.$

Integrating by parts with $u = e^{ax}$, $dv = \cos bx\, dx$, we get

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \int e^{ax} \sin bx\, dx.$$

Integrating by parts again with $u = e^{ax}$, $dv = \sin bx\,dx$, this becomes

$$\int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{b\sin bx + a\cos bx}{b^2}\right) - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx.$$

Transposing the last integral and dividing by $1 + \frac{a^2}{b^2}$ gives

$$\int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{b\sin bx + a\cos bx}{a^2+b^2}\right) + C.$$

125. Reduction Formulas. Integration by parts is often used to make an integral depend on a simpler one and so to obtain a formula by repeated use of which the given integral can be determined.

To illustrate this take

$$\int \sin^n x\,dx,$$

where n is a positive integer. Integrating by parts with

$$u = \sin^{n-1} x, \quad dv = \sin x\,dx,$$

we have

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + \int (n-1)\sin^{n-2} x\cos^2 x\,dx\\ &= -\sin^{n-1} x\cos x\\ &\qquad + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx\\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx\\ &\qquad - (n-1)\int \sin^n x\,dx.\end{aligned}$$

Transposing the last integral and dividing by n, we get

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

This formula expresses the integral of $\sin^n x$ in terms of that of $\sin^{n-2} x$. By repeated application of the formula the result will finally be made to depend on $\int dx$ or $\int \sin x\,dx$ according as n is even or odd.

Example. $\int_0^{\frac{\pi}{2}} \sin^6 x\, dx.$

By the formula just proved

$$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = -\frac{\sin^5 x \cos x}{6}\Bigg|_0^{\frac{\pi}{2}} + \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x\, dx.$$

At the given limits the integrated part is zero. Thus

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^6 x\, dx &= \tfrac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x\, dx \\ &= \tfrac{5}{6}\cdot\tfrac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x\, dx \\ &= \tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\int_0^{\frac{\pi}{2}} dx \\ &= \tfrac{5}{32}\pi.\end{aligned}$$

126. Infinite Values. The definite integral

$$\int_a^b f(x)\, dx \tag{1}$$

has been defined only for a function $f(x)$ which is bounded in the interval (a, b). If the function becomes infinite at a or b or at any value between the limits, the definition in §42 is not applicable.

First suppose that $f(x)$ becomes infinite at $x = b$ but is integrable in the interval (a, z), where z is any number between a and b. If the integral

$$\int_a^z f(x)\, dx$$

tends to a limit when $z \to b$, the integral between a and b is defined as the limit

$$\int_a^b f(x)\, dx = \lim_{z\to b}\int_a^z f(x)\, dx. \tag{2}$$

Similarly, if $f(x)$ becomes infinite at $x = a$,

$$\int_a^b f(x)\, dx = \lim_{z\to a}\int_z^b f(x)\, dx, \tag{3}$$

provided that the limit on the right exists.

In the integral

$$\int_0^1 \frac{dx}{\sqrt{x}},$$

for example, the integrand

$$f(x) = \frac{1}{\sqrt{x}}$$

becomes infinite at the lower limit. Since

$$\int_z^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_z^1 = 2(1 - \sqrt{z})$$

tends to the limit 2 when $z \to 0$, we have by definition

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{z \to 0} \int_z^1 \frac{dx}{\sqrt{x}} = 2.$$

If $f(x)$ becomes infinite at some value c between a and b, the integral from a to b is defined by the equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \tag{4}$$

provided that both limits on the right exist.

Take, for example,

$$\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}}.$$

The function

$$f(x) = \frac{1}{(x-1)^{\frac{2}{3}}}$$

is continuous except at $x = 1$ where it becomes infinite. If z_1 is between 0 and 1, and z_2 is between 1 and 2,

$$\int_0^{z_1} \frac{dx}{(x-1)^{\frac{2}{3}}} = [3(x-1)^{\frac{1}{3}}]_0^{z_1} = 3\,[(z_1 - 1)^{\frac{1}{3}} + 1],$$

$$\int_{z_2}^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = [3(x-1)^{\frac{1}{3}}]_{z_2}^2 = 3\,[1 - (z_2 - 1)^{\frac{1}{3}}].$$

Since both of these tend to limits when z_1 and z_2 tend to 1, the integral from 0 to 2 is the sum of those limits, namely

$$\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = 3 + 3 = 6.$$

In the above discussions it has been assumed that a and b are both finite. If

$$\int_a^b f(x)\,dx$$

tends to a limit when $b \to \infty$, the integral with infinite limit is defined by the equation

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx. \tag{5}$$

Similarly, the integral with lower limit negatively infinite is defined by the equation

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx. \tag{6}$$

For example,

$$\int_0^b e^{-x}\,dx = [-e^{-x}]_0^b = 1 - e^{-b}.$$

Thus

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} (1 - e^{-b}) = 1.$$

In each of the above definitions, (2), (3), (4), (5), (6), the integral on the left is said to *converge*, or to *exist*, if the limit on the right has a definite value, and to *diverge* if that limit does not exist.

All the integrals in the above examples have been convergent. As illustrations of integrals which diverge, take

$$\int_0^1 \frac{dx}{x}, \quad \int_0^\infty \sin x\,dx.$$

The first of these diverges since

$$\int_z^1 \frac{dx}{x} = -\ln z$$

becomes infinite when $z \to 0$. The second diverges since

$$\int_0^b \sin x\,dx = 1 - \cos b$$

does not approach a limit when $b \to \infty$.

127. Tests for Convergence. We shall be concerned only with integrals of functions which between the limits of integration have at most a finite number of discontinuities. From the definitions it follows immediately that, if the indefinite integral

$$\int f(x)\,dx = F(x)$$

of such a function is continuous in the interval of integration and at its limits, the definite integral is convergent and its value is determined by the usual equation

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

When the indefinite integral of a function is not known the convergence or divergence of its definite integral can usually be determined by comparison with that of a simpler function which can be integrated. The principle employed is the following:

Theorem. *Suppose that $p(x)$ is positive and $f(x)$ and $p(x)$ are continuous for all values of x in the interval (a, b) except $x = b$. If*

$$\lim_{x \to b} \frac{f(x)}{p(x)} = k \neq 0, \tag{1}$$

the integrals

$$\int_a^b f(x)\,dx, \quad \int_a^b p(x)\,dx$$

are both convergent or both divergent.

To prove this we note that because of (1) the function $f(x)$ has a fixed algebraic sign when x is sufficiently near b. As z approaches b the integrals

$$\int_a^z f(x)\,dx, \quad \int_a^z p(x)\,dx$$

thus ultimately increase continually or decrease continually. They therefore approach limits or become infinite (§14, (2)). When z is near b any change in the first integral is approximately k times the corresponding change in the second. Thus either integral remains finite if the other does.

To express that two integrals with the same limits both converge or both diverge we may sometimes say that the one integral behaves like the other.

In applying this theorem we can break the interval of integration into a sum of parts and use a different comparison function on each part. The whole integral is then convergent if each part is convergent and divergent if any part is divergent.

Example 1. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x^2)}}.$

On the interval of integration the integrand

$$\frac{1}{\sqrt{x(1-x^2)}}$$

becomes infinite only at the limits $x = 0$, $x = 1$. Near $x = 0$ the integral behaves like

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x},$$

which is convergent. Near $x = 1$ it behaves like

$$\int \frac{dx}{\sqrt{1-x}} = -2\sqrt{1-x},$$

which is also convergent. The given integral is therefore convergent.

Example 2. $\displaystyle\int_0^\infty \frac{dx}{(x^2-1)^{\frac{1}{3}}}.$

This integral must be investigated at $x = 1$ and at the infinite limit. The part of the integral near $x = 1$ behaves like

$$\int \frac{dx}{(x-1)^{\frac{1}{3}}} = \frac{3}{2}(x-1)^{\frac{2}{3}},$$

which is convergent. The part corresponding to very large values of x behaves like

$$\int \frac{dx}{x^{\frac{2}{3}}} = 3x^{\frac{1}{3}},$$

which becomes infinite when $x \to \infty$. The given integral is divergent.

Example 3. $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx.$

We might try

$$p(x) = \frac{1}{x}, \quad f(x) = \frac{\sin x}{x}.$$

But, since

$$\frac{f(x)}{p(x)} = \sin x$$

does not tend to a limit as x tends to infinity, the above theorem is not applicable. Whether this integral converges or diverges cannot be determined by our present methods.

128. Numerical Integration. When the indefinite integral of a function $f(x)$ is unknown or too complicated for convenient use its definite integral

$$\int_a^b f(x)\,dx \tag{1}$$

between given limits can still be computed to any desired accuracy by numerical methods.

One way to do this is to make direct use of the definition of a definite integral as limit of a sum (§42).

Another way is to divide the interval of integration into parts and on each part replace $f(x)$ by an approximating function which can be easily integrated. The parts are usually taken of equal length, and the approximating functions are usually polynomials.

129. Prismoid Formula. Given any second-degree polynomial,

$$f(x) = A + Bx + Cx^2, \tag{1}$$

we can express the constants A, B, C, and so the integral

$$\int_a^b f(x)\,dx,$$

in terms of the values the polynomial takes at three points. If we take $x = a$, $x = b$, and the midpoint $x = \frac{1}{2}(a + b)$ as the three points, the expression obtained for the integral is

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right], \tag{2}$$

as may be directly verified by substituting the above value for $f(x)$ on both sides. In fact by substituting

$$f(x) = A + Bx + Cx^2 + Dx^3$$

this equation can be shown to hold even more generally when $f(x)$ is any third-degree polynomial.

Since equation (2) can be used to determine the volumes of certain solids called prismoids, it is called the *prismoid formula*. When $f(x)$ is a polynomial of not more than the third degree it gives the exact value of the integral on the left. When $f(x)$ is not of this form it can be used to obtain an approximation for the integral, namely the approximation that results when $f(x)$ is replaced by the second-degree polynomial which at a, b, and $\frac{1}{2}(a + b)$ has the

same values as $f(x)$. The accuracy of the approximation is determined by the accuracy with which $f(x)$ can be thus represented.

Example 1. Find the volume of a truncated cone, the radii of its bases being r_1, r_2, and its altitude h.

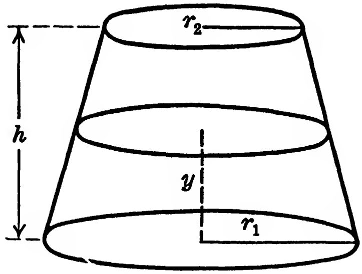

FIGURE 170.

Cross sections of the cone perpendicular to the axis are circles of area

$$A = f(y),$$

which is a quadratic function of the distance y along the axis. The volume

$$v = \int_0^h A\,dy$$

is thus determined exactly by the prismoid formula. Since the cross sections at the ends and middle are circles of radii r_1, r_2, and $\frac{1}{2}(r_1 + r_2)$ the result is

$$v = \frac{h}{6}\left[\pi r_1^2 + 4\pi\left(\frac{r_1 + r_2}{2}\right)^2 + \pi r_2^2\right] = \frac{\pi h}{6}[r_1^2 + (r_1 + r_2)^2 + r_2^2].$$

Example 2. Find an approximate value for

$$\int_1^2 \frac{dx}{x}$$

by the prismoid formula.

Here

$$f(x) = \frac{1}{x}.$$

The values of this function at the limits of integration and the midpoint are

$$f(1) = 1, \quad f(2) = \tfrac{1}{2}, \quad f(\tfrac{3}{2}) = \tfrac{2}{3}.$$

The prismoid formula thus gives

$$\int_1^2 \frac{dx}{x} = \frac{2-1}{6}\left[1 + 4\left(\frac{2}{3}\right) + \frac{1}{2}\right] = 0.694.$$

The answer correct to three decimals is 0.693 (compare §41, example).

130. Simpson's Rule. To obtain a better approximation for a definite integral

$$\int_a^b f(x)\,dx$$

than that given directly by the prismoid formula we can divide the interval of integration into any even number of equal parts, apply the prismoid formula to successive pairs of parts, and add.

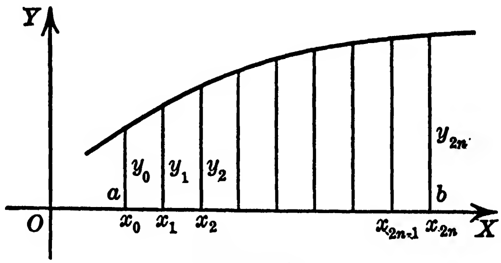

FIGURE 171.

Let $2n$ be the number of parts, each of length

$$h = \frac{b-a}{2n},$$

$x_1, x_2, \cdots, x_{2n-1}$ the points of division arranged in order between $x_0 = a$ and $x_{2n} = b$, and

$$y_i = f(x_i), \quad i = 0, 1, \cdots, 2n,$$

the values of $f(x)$ at those points. Since each of the intervals (a, x_2), (x_2, x_4), etc., is of length $2h$, the prismoid formula gives

$$\int_a^{x_2} f(x)\,dx = \frac{h}{3}\,[y_0 + 4y_1 + y_2],$$

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\,[y_2 + 4y_3 + y_4],$$

. . . .

$$\int_{x_{2n-2}}^{b} f(x)\,dx = \frac{h}{3}\,[y_{2n-2} + 4y_{2n-1} + y_{2n}],$$

whence by addition

$$\int_a^b f(x)\,dx = \frac{h}{3}\,[y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{2n-1} + y_{2n}].$$

This is known as *Simpson's rule.* It gives an approximation for the integral in terms of an odd number of ordinates, the distance between consecutive ordinates being h. The bracket contains a sum of multiples of these ordinates, the first and last coefficients being unity and the others alternately 4 and 2. In general the accuracy of the approximation increases with the number of divisions.

Example. Find π from the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1 + x^2}$$

by dividing the interval into four parts and using Simpson's rule.

The divisions are determined by

$$x = 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1.$$

The corresponding values of

$$y = \frac{1}{1 + x^2}$$

are

$$y = 1, \tfrac{16}{17}, \tfrac{4}{5}, \tfrac{16}{25}, \tfrac{1}{2}.$$

Substituting these and $h = \frac{1}{4}$, Simpson's rule gives

$$\frac{\pi}{4} = \frac{1}{12}\left[1 + 4\left(\frac{16}{17}\right) + 2\left(\frac{4}{5}\right) + 4\left(\frac{16}{25}\right) + \frac{1}{2}\right] = 0.785392,$$

whence

$$\pi = 3.14157.$$

The value correct to 5 decimals is

$$\pi = 3.14159.$$

PROBLEMS

Determine the values of the following integrals:

1. $\int \sqrt{3x + 2}\, dx.$

2. $\int \frac{dx}{(2x - 3)^2}.$

3. $\int \frac{dx}{\sqrt{a + bx}}.$

4. $\int (\sqrt{a} - \sqrt{x}\,)^2\, dx.$

5. $\int \frac{x\, dx}{\sqrt{a^2 - x^2}}.$

6. $\int x^2\sqrt{x^3 + 1}\, dx.$

7. $\int x^{\frac{1}{2}}\sqrt{x^{\frac{3}{2}} - 1}\, dx.$

8. $\int \frac{2x\, dx}{(3x^2 - 2)^2}.$

9. $\int \frac{dx}{4x + 3}.$

10. $\int \frac{x\, dx}{2x^2 - 1}.$

11. $\int \frac{x^2\,dx}{1+x^3}$.

12. $\int \frac{(2x+3)\,dx}{x^2+3x+2}$.

13. $\int 2\cos(4x+1)\,dx$.

14. $\int \sin\frac{3x-2}{5}\,dx$.

15. $\int x\sin(x^2+1)\,dx$.

16. $\int x\cos(ax^2)\,dx$.

17. $\int \sec^2\frac{\theta}{2}\,d\theta$.

18. $\int \frac{d\theta}{\cos^2 3\theta}$.

19. $\int \frac{d\theta}{\sin^2 2\theta}$.

20. $\int x\csc^2(x^2)\,dx$.

21. $\int \tan\frac{3x+4}{5}\,dx$.

22. $\int \frac{dx}{\tan 2x}$.

23. $\int \frac{\cos 2x}{\sin 2x}\,dx$.

24. $\int \sec 5\theta \tan 5\theta\,d\theta$.

25. $\int \frac{\cos x\,dx}{\sin^2 x}$.

26. $\int \sec\theta(\sec\theta+\tan\theta)\,d\theta$.

27. $\int (\csc\theta-\cot\theta)\csc\theta\,d\theta$.

28. $\int \frac{1+\sin 2x}{\cos^2 2x}\,dx$.

29. $\int \frac{dx}{\cos 3x}$.

30. $\int \frac{dx}{\sin(3x+2)}$.

31. $\int \frac{1+\sin x}{\cos x}\,dx$.

32. $\int (1+\sec\theta)^2\,d\theta$.

33. $\int \frac{\csc^2 x\,dx}{1+2\cot x}$.

34. $\int \frac{dx}{x\ln x}$.

35. $\int \frac{dt}{\sqrt{2-t^2}}$.

36. $\int \frac{dx}{\sqrt{3-4x^2}}$.

37. $\int \frac{(2x+3)\,dx}{\sqrt{4-x^2}}$.

38. $\int \frac{dx}{x^2+5}$.

39. $\int \frac{dx}{4x^2+3}$.

40. $\int \frac{dx}{2x^2+3}$.

41. $\int \frac{dx}{x\sqrt{x^2-4}}$.

42. $\int \frac{dx}{x\sqrt{4x^2-9}}$.

43. $\int \frac{dx}{x\sqrt{5x^2-1}}$.

44. $\int \frac{dx}{\sqrt{x^2+4}}$.

45. $\int \frac{dx}{\sqrt{4x^2+3}}$.

46. $\int \frac{dx}{\sqrt{2x^2+5}}$.

47. $\int \frac{dx}{x^2-16}$.

48. $\int \frac{dx}{4x^2-9}$.

49. $\int \frac{dx}{2x^2-3}$.

50. $\int e^{-3x}\,dx$.

51. $\int \frac{dx}{e^{2x}}$.

52. $\int (e^x-e^{-x})^2\,dx$.

53. $\int xe^{-x^2}\,dx$.

54. $\int \frac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}}$.

55. $\int \frac{\cos\theta\,d\theta}{\sqrt{2-\sin^2\theta}}$.

56. $\int \frac{x\,dx}{\sqrt{a^4-x^4}}$.

57. $\int \frac{e^{2x}\,dx}{1+e^{2x}}$.

58. $\int \frac{e^x\,dx}{1+e^{2x}}$.

59. $\int \frac{\cos\theta\,d\theta}{2+\sin^2\theta}$.

60. $\int \frac{dt}{\sqrt{e^{2t}-1}}$.

61. $\int \sin^3 x\cos x\,dx$.

62. $\int \cos^4 5x\sin 5x\,dx$.

63. $\int (\cos\theta+\sin\theta)^2\,d\theta$.

64. $\int \sin^3 x\,dx$.

65. $\int \cos^3 2x\,dx$.

66. $\int \sin^3 x\cos^2 x\,dx$.

67. $\int \cos^5 x\,dx$.

68. $\int \sin^3 4\theta\cos^3 4\theta\,d\theta$.

69. $\int \frac{\sin^3 x\,dx}{1-\cos x}$.

70. $\int \frac{\cos^2 x\,dx}{\sin x}$.

71. $\int \frac{\cos^3 x\,dx}{\sin x}$.

72. $\int \tan^2 x\sec^2 x\,dx$.

73. $\int \sec^2 x\tan^3 x\,dx$.

74. $\int \frac{\sin x\,dx}{\cos^3 x}$.

75. $\int \frac{\sin^2 x\,dx}{\cos^4 x}$.

76. $\int \sec^4 x\,dx$.

77. $\int \tan^2 x\,dx$.

78. $\int (1+\cot\theta)^2\,d\theta$.

79. $\int \sec^4 x\tan^3 x\,dx$.

80. $\int \csc^6 x\,dx$.

81. $\int \frac{\cos^2 t\,dt}{\sin^6 t}$.

82. $\int \tan\theta\csc\theta\,d\theta$.

83. $\int \cos^2 4x\,dx$.

84. $\int (1+\cos\theta)^2\,d\theta$.

85. $\int (1-\sin x)^3\,dx$.

86. $\int \sin^4 x\,dx$.

87. $\int \sin^2 2x\cos^2 2x\,dx$.

88. $\int \sin^4\theta\cos^2\theta\,d\theta$.

89. $\int \cos^6 x\,dx$.

90. $\int \cos x\sin 2x\,dx$.

91. $\int \sin x\cos 3x\,dx$.

92. $\int \sin 2x\sin 3x\,dx$.

93. $\int \cos 2x\cos 4x\,dx$.

94. $\int \sin^2 2x\cos 3x\,dx$.

95. $\int \sec x\csc x\,dx$.

96. $\int \frac{dx}{1-\cos x}$.

97. $\int \sqrt{1+\cos\theta}\,d\theta$.

98. $\int (1+\cos\theta)^{\frac{3}{2}}\,d\theta$.

99. $\displaystyle\int \frac{dx}{\sqrt{2 + 2x - x^2}}.$

100. $\displaystyle\int \frac{dx}{\sqrt{1 + 4x - 4x^2}}.$

101. $\displaystyle\int \frac{dx}{\sqrt{2 + 6x - 3x^2}}.$

102. $\displaystyle\int \frac{dx}{x^2 + 6x + 13}.$

103. $\displaystyle\int \frac{dx}{2x^2 - 4x + 6}.$

104. $\displaystyle\int \frac{dx}{(x - 1)\sqrt{x^2 - 2x - 3}}.$

105. $\displaystyle\int \frac{dx}{(2x - 3)\sqrt{x^2 - 3x + 1}}.$

106. $\displaystyle\int \frac{dx}{(2x - 1)\sqrt{x^2 - x}}.$

107. $\displaystyle\int \frac{(2x + 3)\,dx}{x^2 + 2x - 3}.$

108. $\displaystyle\int \frac{(x + 1)\,dx}{x^2 + 2x - 3}.$

109. $\displaystyle\int \frac{(x - 1)\,dx}{4x^2 - 4x + 2}.$

110. $\displaystyle\int \frac{x\,dx}{x^2 + 2x + 2}.$

111. $\displaystyle\int \frac{(4x + 1)\,dx}{\sqrt{1 + 4x - 4x^2}}.$

112. $\displaystyle\int \frac{(3x - 2)\,dx}{\sqrt{x^2 + 2x + 3}}.$

113. $\displaystyle\int \frac{(2x - 1)\,dx}{\sqrt{3x^2 - 6x - 1}}.$

114. $\displaystyle\int \frac{(x - 1)\,dx}{(x^2 - 2x + 3)^{\frac{3}{2}}}.$

115. $\displaystyle\int \frac{e^x\,dx}{e^{2x} + 2e^x + 3}.$

116. $\displaystyle\int \frac{x^2\,dx}{x^2 + x - 6}.$

117. $\displaystyle\int \frac{x + 2}{x^2 + x}\,dx.$

118. $\displaystyle\int \frac{(x^3 + x^2)\,dx}{x^2 - 3x + 2}.$

119. $\displaystyle\int \frac{dx}{x^3 - x}.$

120. $\displaystyle\int \frac{(x - 3)\,dx}{x^3 + 3x^2 + 2x}.$

121. $\displaystyle\int \frac{x^3 + 1}{x^3 - x^2}\,dx.$

122. $\displaystyle\int \frac{x\,dx}{(x + 1)^2}.$

123. $\displaystyle\int \frac{(x + 2)\,dx}{x^2 - 4x + 4}.$

124. $\displaystyle\int \frac{(3x + 2)\,dx}{x^3 - 2x^2 + x}.$

125. $\displaystyle\int \frac{8\,dx}{x^4 - 2x^3}.$

126. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}.$

127. $\displaystyle\int \frac{(1 - x^3)\,dx}{x(x^2 + 1)}.$

128. $\displaystyle\int \frac{(x - 1)\,dx}{(x + 1)(x^2 + 1)}.$

129. $\displaystyle\int \frac{4x\,dx}{x^4 - 1}.$

130. $\displaystyle\int \frac{3(x + 1)\,dx}{x^3 - 1}.$

131. $\displaystyle\int \frac{(x^4 + x)\,dx}{x^4 - 4}.$

132. $\displaystyle\int \frac{x^2\,dx}{(x^2 + 1)(x^2 + 2)}.$

133. $\displaystyle\int \frac{3\,dx}{x^4 + 5x^2 + 4}.$

134. $\displaystyle\int \frac{(x - 1)\,dx}{(x^2 + 1)(x^2 - 2x + 3)}.$

135. $\displaystyle\int \frac{x^3\,dx}{(x^2 + 4)^2}.$

136. $\displaystyle\int \frac{(x^4 + 1)\,dx}{x(x^2 + 1)^2}.$

137. $\displaystyle\int \frac{(x^2 + 1)\,dx}{(x^2 - 2x + 3)^2}.$

138. $\displaystyle\int \frac{1 + \sqrt{x}}{1 - \sqrt{x}}\,dx.$

139. $\displaystyle\int \frac{x\,dx}{\sqrt{x + 1}}.$

140. $\displaystyle\int x\sqrt{x - a}\,dx.$

141. $\displaystyle\int \frac{\sqrt{x+2}}{x+3}\,dx.$

142. $\displaystyle\int \frac{dx}{x\sqrt{x-1}}.$

143. $\displaystyle\int \frac{dx}{x+2\sqrt{x-1}}.$

144. $\displaystyle\int \frac{x^{\frac{1}{4}}\,dx}{1+x^{\frac{1}{2}}}.$

145. $\displaystyle\int \frac{dx}{x^{\frac{1}{2}}-x^{\frac{1}{3}}}.$

146. $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx.$

147. $\displaystyle\int \frac{x^3\,dx}{\sqrt{a^2-x^2}}.$

148. $\displaystyle\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}.$

149. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}}.$

150. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}}.$

151. $\displaystyle\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}.$

152. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}}.$

153. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-a^2}}.$

154. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx.$

155. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^3}\,dx.$

156. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-a^2}}.$

157. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx.$

158. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}}.$

159. $\displaystyle\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}.$

160. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^{\frac{3}{2}}}.$

161. $\displaystyle\int \frac{x^2\,dx}{(a^2+x^2)^2}.$

162. $\displaystyle\int x\cos x\,dx.$

163. $\displaystyle\int x\sin x\,dx.$

164. $\displaystyle\int x\sec^2 x\,dx.$

165. $\displaystyle\int x\sec x\tan x\,dx.$

166. $\displaystyle\int x^2e^x\,dx.$

167. $\displaystyle\int x^4\ln x\,dx.$

168. $\displaystyle\int x^3e^{x^2}\,dx.$

169. $\displaystyle\int \sin^{-1}x\,dx.$

170. $\displaystyle\int \tan^{-1}x\,dx.$

171. $\displaystyle\int \ln\left(x+\sqrt{x^2+a^2}\right)dx.$

172. $\displaystyle\int x\sec^{-1}x\,dx.$

173. $\displaystyle\int (x-1)^2\sin x\,dx.$

174. $\displaystyle\int \sqrt{x^2-a^2}\,dx.$

175. $\displaystyle\int \sqrt{x^2+a^2}\,dx.$

176. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2-a^2}}.$

177. $\displaystyle\int e^{2x}\sin 3x\,dx.$

178. $\displaystyle\int e^{-x}\cos x\,dx.$

179. $\displaystyle\int \sin 3x\cos 2x\,dx.$

180. Derive the formula

$$\int \sec^n x = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx,$$

and use it to integrate

$$\int \sec^3 x \, dx.$$

181. Derive the formula

$$\int (a^2 - x^2)^n \, dx = \frac{x(a^2 - x^2)^n}{2n+1} + \frac{2a^2 n}{2n+1} \int (a^2 - x^2)^{n-1} \, dx,$$

and use it to integrate

$$\int_0^a (a^2 - x^2)^{\frac{5}{2}} \, dx.$$

182. By solving the equation of the preceding problem for the integral on the right and changing notation, derive the formula

$$\int (a^2 - x^2)^n \, dx = -\frac{x(a^2 - x^2)^{n+1}}{2a^2(n+1)} + \frac{2n+3}{2a^2(n+1)} \int (a^2 - x^2)^{n+1} \, dx,$$

and use it to integrate

$$\int \frac{dx}{(a^2 - x^2)^{\frac{5}{2}}}.$$

By finding the indefinite integral determine whether each of the following integrals is convergent, and if convergent find the value of the integral:

183. $\displaystyle\int_0^1 \frac{dx}{x^{\frac{2}{3}}}.$

184. $\displaystyle\int_0^a \frac{x \, dx}{(a^2 - x^2)^{\frac{3}{2}}}.$

185. $\displaystyle\int_{-1}^1 \frac{dx}{x^2}.$

186. $\displaystyle\int_{-1}^1 \frac{dx}{x^{\frac{4}{5}}}.$

187. $\displaystyle\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}.$

188. $\displaystyle\int_0^\infty e^{-3x} \, dx.$

189. $\displaystyle\int_1^\infty \frac{dx}{x \ln x}.$

190. $\displaystyle\int_0^\infty \sin^2 x \, dx.$

Determine whether the following integrals converge:

191. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x - x^3}}.$

192. $\displaystyle\int_{-2}^2 \frac{dx}{(x^2 - 1)^{\frac{2}{3}}}.$

193. $\displaystyle\int_0^2 \frac{dx}{(x^2 - 1)\sqrt{x}}.$

194. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin \theta}}.$

195. $\displaystyle\int_{-1}^1 \frac{dx}{(1 - x^2)^{\frac{4}{3}}}.$

196. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4 - 1}}.$

197. $\displaystyle\int_0^\infty \frac{x \, dx}{\sqrt{x^3 + 1}}.$

198. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^3 + x}}.$

199. Show that the prismoid formula gives the correct volume for each of the following solids: (1) sphere, (2) cone, (3) cylinder, (4) pyramid, (5) segment of a sphere, (6) ellipsoid generated by rotating an ellipse about its major axis.

In each of the following examples compare the value given by the prismoid formula with that obtained by integration:

200. $\int_0^1 \sqrt{x}\,dx.$

201. $\int_{-1}^1 \frac{dx}{(x+4)^2}.$

202. $\int_0^{\frac{\pi}{2}} \sin x\,dx.$

203. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2 x\,dx.$

Compute each of the following integrals by Simpson's rule with four intervals:

204. $\int_{-2}^2 \frac{dx}{1+x^4}.$

205. $\int_{-1}^3 \sqrt{1+x^3}\,dx.$

206. $\int_{-3}^3 \sqrt[3]{1+x^2}\,dx.$

207. $\int_0^{\pi} \sqrt{\sin x}\,dx.$

208. $\int_0^{\frac{\pi}{3}} \sqrt{2+\cos x}\,dx.$

209. $\int_0^4 e^{-x^2}\,dx.$

210. Two parabolas with axes parallel to OY intersect and are tangent at $x = \frac{1}{2}(a+b)$. A curve $y = f(x)$ coincides with one of the parabolas between $x = a$ and $x = \frac{1}{2}(a+b)$ and with the other between $x = \frac{1}{2}(a+b)$ and $x = b$. Show that

$$\int_a^b f(x)\,dx$$

is represented exactly by the prismoid formula.

CHAPTER XII

FURTHER APPLICATIONS OF INTEGRATION

131. Area of a Surface of Revolution. The surface generated by rotating an arc AB of a plane curve about an axis in its plane is called a *surface of revolution.*

To define the area of such a surface divide the arc AB into parts by points arranged in order from A to B, connect consecutive

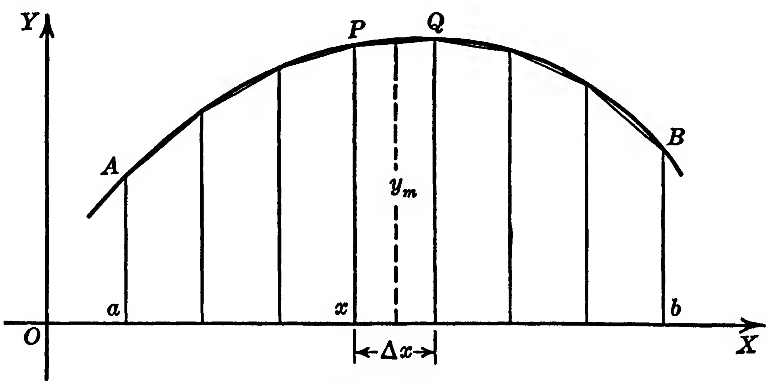

FIGURE 172.

points by chords, and determine the sum of the conical areas generated by rotating these chords about the axis. If this sum approaches a limit as the number of divisions is increased and all the chords tend to zero, that limit is defined as the area of the surface.

Suppose, for example, that the arc of the curve

$$y = f(x)$$

between $x = a$ and $x = b$ is rotated about the x-axis. We assume that y is positive and

$$\frac{dy}{dx} = f'(x)$$

continuous on the interval (a, b). Divide the arc into parts, and let $P(x, y)$, $Q(x + \Delta x, y + \Delta y)$ be consecutive points of division.

The chord joining these points has the ordinate

$$y_m = y + \tfrac{1}{2}\,\Delta y$$

at its midpoint. The area generated by rotating this chord about the x-axis is

$$2\pi y_m\overline{PQ} = 2\pi(y + \tfrac{1}{2}\,\Delta y)\overline{PQ}.$$

Except for infinitesimals of higher order than Δx, this is equivalent to

$$2\pi y\,\Delta s,$$

where Δs is the arc PQ. As the chords tend to zero the sum of the areas they generate thus tends to the limit

$$S = \lim_{\Delta s \to 0} \Sigma 2\pi y\,\Delta s = \int 2\pi y\,ds, \tag{1}$$

which is therefore the area of the surface of revolution.

If the curve is rotated about some other axis, this equation is replaced by

$$S = \int 2\pi r\,ds, \tag{2}$$

where r is the distance of ds from the axis.

Since the limits of integration depend on the variable actually used, no limits have been indicated in these formulas. To evaluate the integrals, y and ds or r and ds must be expressed in terms of a single variable t. The limits are then the values this variable takes at the ends of the arc. Also it has been assumed that ds is positive. The variable t and its limits must therefore be such that, as t varies from the lower to the upper limit,

$$ds = \frac{ds}{dt}\,dt$$

never becomes negative.

Example 1. Find the area of the surface generated by rotating an ellipse about its major axis.

Take as equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and assume the major axis to be the x-axis. Then $a > b$ and

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \frac{b\sqrt{a^4 - (a^2 - b^2)x^2}\,dx}{a^2 y}.$$

The entire surface is generated by the part of the curve above the x-axis between $x = -a$ and $x = a$. Its area is therefore

$$S = \int 2\pi y\, ds = \frac{2\pi b}{a^2} \int_{-a}^{a} \sqrt{a^4 - (a^2 - b^2)x^2}\, dx.$$

Integrating as in Example 3, §124, and expressing the result in terms of the eccentricity

$$e = \frac{\sqrt{a^2 - b^2}}{a},$$

we obtain

$$S = 2\pi b \left[b + \frac{a}{e} \sin^{-1} e \right].$$

Example 2. Find the area of the surface generated by rotating the curve

$$r = a(2 + \cos\theta)$$

about the initial line.

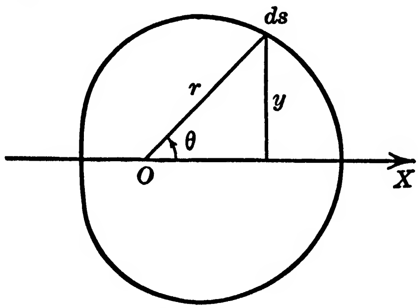

FIGURE 173.

In this case

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = a\sqrt{5 + 4\cos\theta}\, d\theta,$$

$$y = r\sin\theta = a(2 + \cos\theta)\sin\theta.$$

The entire area is generated by the part of the curve above the x-axis between $\theta = 0$ and $\theta = \pi$. Its area is therefore

$$S = \int 2\pi y\, ds = 2\pi a^2 \int_0^{\pi} (2 + \cos\theta)\sqrt{5 + 4\cos\theta}\, \sin\theta\, d\theta.$$

To evaluate this we substitute

$$5 + 4\cos\theta = z^2.$$

When θ varies from 0 to π, z varies from 3 to 1 and the integral becomes

$$S = \frac{\pi a^2}{4} \int_1^3 (3 + z^2)z^2\, dz = \frac{93}{5}\pi a^2.$$

132. Area of a Cylindrical Surface. The locus of lines parallel to a fixed direction and touching a fixed curve is called a *cylindrical surface.*

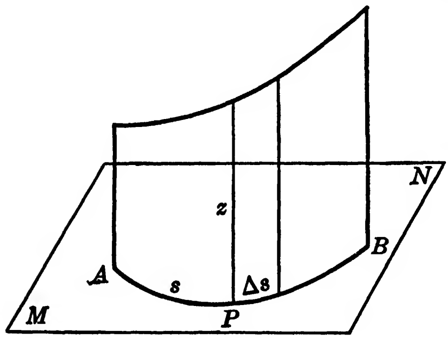

FIGURE 174.

We shall determine the area of the cylindrical surface generated by perpendiculars to a plane MN at points of an arc AB. Let s be the distance measured along AB to the variable point P, and z the length of the perpendicular at P. We assume that AB is a smooth curve and z a continuous function of s. The part of the cylindrical surface between the perpendiculars at s and $s + \Delta s$ is then approximately a rectangle of area $z\ \Delta s$, and the required area is

$$S = \lim_{\Delta s \to 0} \Sigma z\ \Delta s = \int z\, ds.$$

To evaluate this integral z and ds must be expressed in terms of a single variable and the integral taken between the limiting values of that variable.

Example. Find the area cut from the lateral surface of a right circular cylinder by a plane through the center of the base making the angle α with the base.

We are to determine the area cut from the lateral surface of the cylinder (Figure 175) by the plane ABC. Let P be any point on the arc forming the base of the required area, O the center, and

$$\theta = \angle AOP.$$

The height of the perpendicular at P is

$$PQ = MP \tan\alpha = a \sin\theta \tan\alpha,$$

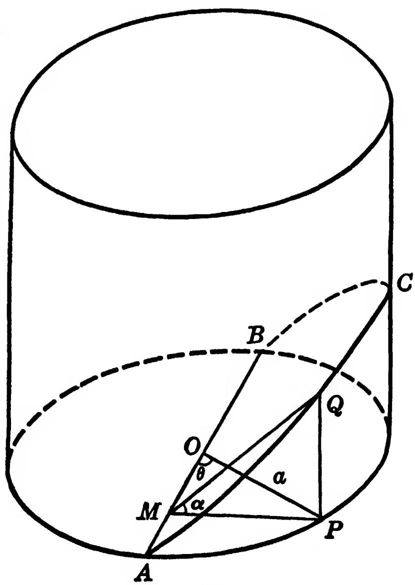

FIGURE 175.

where a is the radius of the cylinder. Along the arc AB,

$$ds = a\,d\theta,$$

and the whole arc is described when θ varies from 0 to π. The required area is therefore

$$\int_0^\pi a^2 \sin\theta \tan\alpha\, d\theta = 2a^2 \tan\alpha.$$

133. Mean Value of a Function. Between $x = a$ and $x = b$ let

$$y = f(x)$$

be a continuous function of x. Divide the interval from a to b into n equal parts

$$\Delta x = \frac{b - a}{n}$$

and let $y_1, y_2, \cdots, y_n$ be the values of y at $x = a$ and the points of division. The average value of these n numbers is

$$\frac{y_1 + y_2 + y_3 + \cdots + y_n}{n}.$$

Multiplying numerator and denominator by Δx, this fraction becomes

$$\frac{y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x}{n\,\Delta x} = \frac{\sum_a^b y\,\Delta x}{\sum_a^b \Delta x} = \frac{\sum_a^b f(x)\,\Delta x}{\sum_a^b \Delta x}.$$

As n increases, this average value tends to the limit

$$\frac{\int_a^b f(x)\,dx}{\int_a^b dx},$$

which is called the *mean value* or *average value* of $f(x)$ with respect to x between $x = a$ and $x = b$.

This may be regarded as a sort of average of all the ordinates of the curve $y = f(x)$ between $x = a$ and $x = b$, but it should be noted that the result depends on the variable x in terms of which y is expressed. In the above discussion we have taken ordinates at equal intervals along the x-axis and have obtained as final result

the average value of y with respect to x. If y is also expressible in terms of some other variable t and we take ordinates at equal intervals Δt, the resulting average with respect to t need not have the same value.

Example. Find the average value of

$$y = \frac{a^3}{a^2 + x^2}$$

between $x = -a$ and $x = a$, (1) if the average is taken with respect to x, and (2) if $x = a \tan \theta$ and the average is taken with respect to θ.

If the average is taken with respect to x, its value is

$$\frac{\int_{-a}^{a} y\,dx}{\int_{-a}^{a} dx} = \frac{\int_{-a}^{a} \frac{a^3}{a^2 + x^2}\,dx}{2a} = \frac{\pi}{4}\,a.$$

If the average is taken with respect to θ, the result is

$$\frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y\,d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta} = \frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a \cos^2\theta\,d\theta}{\frac{\pi}{2}} = \frac{\pi + 2}{2\pi}\,a.$$

134. Moment. If a force F acts in a plane perpendicular to an axis AB (Figure 176), its *moment*, or *torque*, about the axis is the product

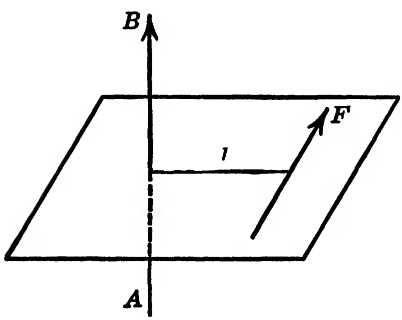

FIGURE 176.

$$M = Fl,$$

where l is the lever arm, or perpendicular distance, from the axis to the line of the force. If a body is acted on by several forces, the total moment about a given axis is the sum of the moments of the separate forces, a moment being considered positive when the force tends to produce rotation in one direction about the axis and negative when it tends to produce rotation in the other direction.

The term moment is also applied to quantities other than forces, the moment being the product of the magnitude of the quantity

by the distance of its position or point of application from a line or a plane.

To find the moment of a plane area, length, or mass, divide it into small parts such that the points of each differ only infinitesimally in distance from an axis in the plane. Multiply each part by the distance of one of its points from the axis, the distance being considered positive for points on one side of the axis, negative for points on the other side. The limit approached by the sum of these products when the parts are taken smaller and smaller is called the moment of the length, area, or mass with respect to the axis.

Similarly, to find the moment of a length, area, volume, or mass in space with respect to a plane, we divide it into elements whose points differ only infinitesimally in distance from the plane and multiply each element by the distance of one of its points from the plane, these distances being considered positive for points on one side of the plane and negative for points on the other side. The moment with respect to the plane is the limit approached by the sum of these products when the elements are taken smaller and smaller.

Example 1. A rectangular floodgate is 10 feet broad and 6 feet deep. Find the moment of water pressure about the base line when the water level is at the top of the gate.

Divide the rectangle into strips of width Δh by horizontal lines. Neglecting infinitesimals of order higher than Δh, the force due to water pressure on the strip between the depths h, $h + \Delta h$ is

$$wh \cdot 10\,\Delta h,$$

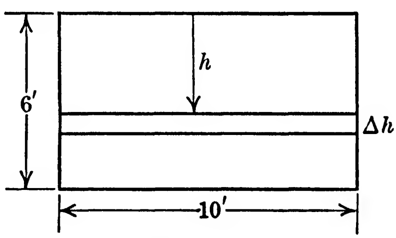

FIGURE 177.

w being the weight of a cubic foot of water. About the base line this force has the lever arm

$$l = 6 - h$$

and moment

$$10wh(6 - h)\,\Delta h$$

approximately. Thus the total moment is

$$\lim_{\Delta h \to 0} \Sigma 10wh(6 - h)\,\Delta h = \int_0^6 10wh(6 - h)\,dh = 360w.$$

Example 2. In the preceding problem find the moment of the area about the base line.

The strip between the depths h, $h + \Delta h$ has the area

$$\Delta A = 10\,\Delta h.$$

All points of this strip are at distance approximately

$$l = 6 - h$$

from the base line. Thus the moment of the area about the base line is

$$\lim_{\Delta h \to 0} \Sigma 10(6 - h)\,\Delta h = \int_0^6 10(6 - h)\,dh = 180.$$

135. Center of Gravity of a Plane Distribution, Centroid. The center of gravity of a system of masses in a plane may be defined as the point at which these masses could be concentrated without changing their total moment with respect to any axis in the plane.

Proof that there is such a point may be left as an exercise. An important consequence is the fact that in finding the moment of a body about an axis it may be cut into parts and each part considered as concentrated at its own center of gravity.

Let $C(\bar{x}, \bar{y})$ be the center of gravity of a continuous mass. This may be a thin wire bent around a curve or a thin plate overlying an area. Cut the mass into pieces, and let x, y be the coördinates of the center of gravity of the piece of mass Δm. The moment of this piece about the x-axis is $y\,\Delta m$, and the total moment of the whole body about the axis is

$$\lim_{\Delta m \to 0} \Sigma y\,\Delta m = \int y\,dm.$$

The mass of the body is

$$M = \int dm.$$

If this were concentrated at the center of gravity, its moment with respect to the x-axis would be $M\bar{y}$. By definition we thus have

$$M\bar{y} = \int y\,dm,$$

whence

$$\bar{y} = \frac{\int y\,dm}{\int dm}. \tag{1}$$

Similarly,

$$\bar{x} = \frac{\int x\,dm}{\int dm}. \tag{2}$$

These formulas show that the rectangular coördinates of the center of gravity are the mean coördinates, the average being taken with respect to the mass.

For a thin wire (Figure 178) we take

$$dm = \rho\,ds, \tag{3}$$

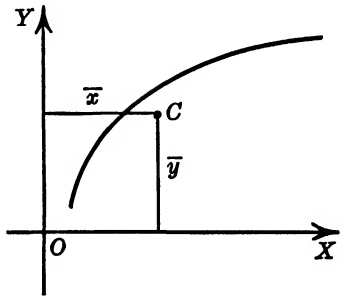

FIGURE 178.

where ρ is the mass per unit length and ds is the element of arc. In particular, if ρ is constant, it may be canceled from numerator and denominator, leaving

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds} \tag{4}$$

as the center of gravity of a thin wire of constant cross section and density.

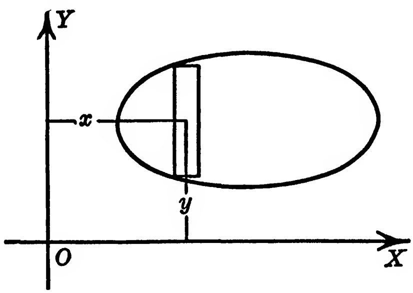

FIGURE 179.

For a thin plate (Figure 179) we take

$$dm = \rho\,dA, \tag{5}$$

where ρ is the mass per unit area and dA is the element of area.

In particular, if ρ is constant, it may be canceled, leaving

$$\bar{x} = \frac{\int x\,dA}{\int dA}, \quad \bar{y} = \frac{\int y\,dA}{\int dA} \tag{6}$$

as the center of gravity of a thin plate of constant thickness and density.

The term center of gravity has its origin in the fact that if the parts of a body are subject to parallel forces proportional to their masses (as under gravity) the resultant of these forces passes through the center of gravity. Expressions (4) and (6), however, also occur frequently in discussions that do not involve mass or force. In such cases (4) is usually called the *centroid* of the arc and (6) the *centroid* of the area, but the expressions center of gravity of arc and center of gravity of area are also frequently used even when there is no association with gravity.

In these integrals x, y are the coördinates of the center of gravity or centroid of the element dm or dA. Before integration all quantities must be expressed in terms of a single variable and proper limits inserted.

If a body is symmetrical with respect to an axis, its moment with respect to that axis is zero. The center of gravity is thus on an axis of symmetry.

Example 1. Find the center of gravity of a thin wire of uniform cross section and density bent into a quadrant of a circle of radius a.

Taking the axes as in Figure 180, $x^2 + y^2 = a^2$, and

$$ds = \sqrt{dx^2 + dy^2} = \frac{a}{y}\,dx.$$

Hence

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^a a\,dx}{\frac{\pi a}{2}} = \frac{2a}{\pi}. \tag{7}$$

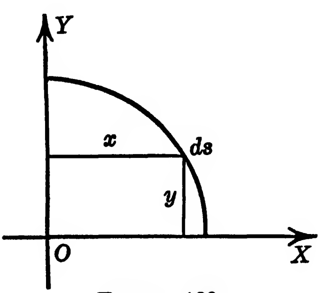

FIGURE 180.

It is evident from the symmetry of the figure that $\bar{x}$ has the same value.

Example 2. Find the centroid of the area of a quadrant of a circle of radius a.

Take the element of area perpendicular to OX (Figure 181). If x, y are the coördinates of the point P at the top of this element, its centroid is $(x + \frac{1}{2}dx, \frac{1}{2}y)$ and its moment with respect to the x-axis is

$$\frac{y}{2}\,dA = \frac{1}{2}y^2\,dx.$$

The moment of the entire area about the x-axis is then

$$\int \tfrac{1}{2}y^2\,dx = \int_0^a \tfrac{1}{2}(a^2 - x^2)\,dx = \tfrac{1}{3}a^3. \tag{8}$$

Thus

$$\bar{y} = \frac{\displaystyle\int \tfrac{1}{2}y\,dA}{\displaystyle\int dA} = \frac{\frac{1}{3}a^3}{\frac{1}{4}\pi a^2} = \frac{4a}{3\pi}, \tag{9}$$

and it is evident that $\bar{x}$ has the same value.

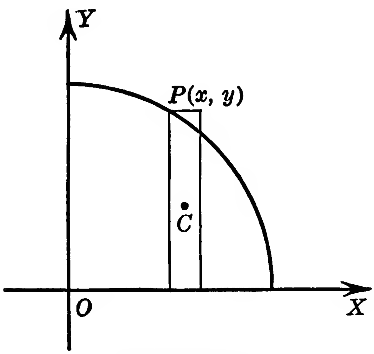

FIGURE 181.

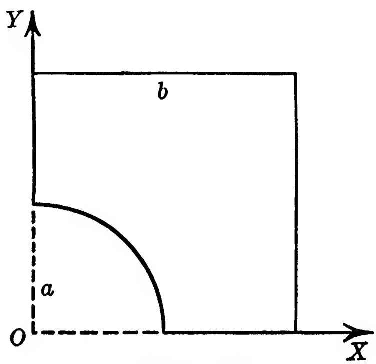

FIGURE 182.

Example 3. A square plate of uniform thickness and density has a quadrant of a circle cut from one corner. Find its center of gravity.

Let b be the side of the square, a the radius of the circular quadrant, and take the coördinate axes as indicated in Figure 182. In the formula

$$\bar{y} = \frac{\displaystyle\int y\,dA}{\displaystyle\int dA}$$

the denominator is the difference between the area of a square and a quadrant of a circle and the numerator is the difference of their moments about the x-axis. Since the moment of the square is

$$b^2 \cdot \frac{b}{2} = \frac{b^3}{2},$$

and equation (8) of the preceding example gives

$$\tfrac{1}{3}a^3$$

as that of the circular quadrant, we have

$$\bar{y} = \frac{\frac{1}{2}b^3 - \frac{1}{3}a^3}{b^2 - \frac{\pi}{4}a^2} = \bar{x}$$

as the coördinates of the center of gravity of the plate.

Example 4. Find the centroid of the area enclosed by the cardioid $r = a(1 + \cos\theta)$.

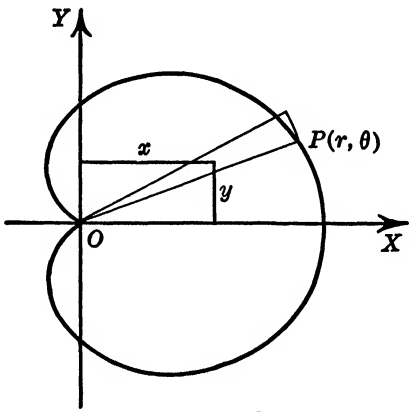

FIGURE 183.

Use the polar element of area

$$dA = \tfrac{1}{2}r^2\,d\theta.$$

This element is approximately a triangle with centroid two-thirds of the way from the vertex to the base. The coördinates of this point are thus approximately

$$x = \tfrac{2}{3}r\cos\theta, \quad y = \tfrac{2}{3}r\sin\theta,$$

r, θ being the coördinates of the point $P(r, \theta)$ on the cardioid (Figure 183). Hence

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^{2\pi} \frac{1}{3}a^3(1+\cos\theta)^3\cos\theta\,d\theta}{\int_0^{2\pi}\frac{1}{2}a^2(1+\cos\theta)^2\,d\theta} = \frac{5}{6}a.$$

By symmetry it is clear that $\bar{y} = 0$.

136. Center of Gravity in Space. The center of gravity of a system of masses in space may be defined as the point at which these masses could be concentrated without changing their total moment with respect to any plane.

Thus to find the center of gravity of a continuous mass cut it into slices of mass Δm. If (x, y, z) is the center of gravity of such a slice, its moment with respect to the xy-plane is $z\,\Delta m$ and the moment of the whole mass is

$$\lim_{\Delta m \to 0} \Sigma z\,\Delta m = \int z\,dm.$$

If the whole mass were concentrated at the center of gravity $(\bar{x}, \bar{y}, \bar{z})$, its moment with respect to the xy-plane would be

$$\bar{z}\int dm.$$

By definition we thus have

$$\bar{z}\int dm = \int z\,dm,$$

whence

$$\bar{z} = \frac{\int z\,dm}{\int dm}. \tag{1}$$

Similarly,

$$\bar{x} = \frac{\int x\,dm}{\int dm}, \quad \bar{y} = \frac{\int y\,dm}{\int dm}. \tag{2}$$

To find the center of gravity, or centroid, of an arc, surface, or volume it is merely necessary to replace the element of mass in these formulas by the appropriate element of arc, area, or volume and insert the proper limits.

Example 1. Find the center of gravity of a homogeneous hemisphere of radius a.

By symmetry the center of gravity is on the radius perpendicular to the plane face. Cut the hemisphere into slices by planes perpendicular to this radius (Figure 184). The slice between the planes at distances

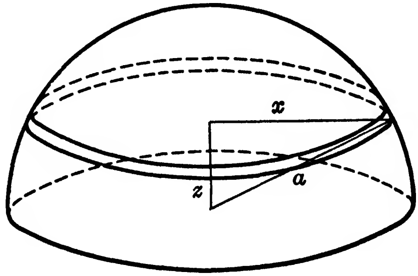

FIGURE 184.

z, $z + dz$ from the plane face has approximately the volume

$$dv = \pi x^2\, dz = \pi(a^2 - z^2)\, dz$$

and approximately the moment

$$z\, dv = \pi z(a^2 - z^2)\, dz$$

with respect to the plane $z = 0$. The center of gravity of the hemisphere is thus on the radius perpendicular to the plane face at distance

$$\bar{z} = \frac{\int z\, dv}{\int dv} = \frac{\int_0^a \pi z(a^2 - z^2)\, dz}{\frac{2}{3}\pi a^3} = \frac{3}{8}a$$

from the center.

Example 2. Find the center of gravity of a thin hemispherical shell of radius a and uniform thickness and density.

Divide the shell (Figure 184) into strips by planes parallel to its base. Each of these strips may be considered as generated by rotating an arc Δs of the circle

$$x^2 + z^2 = a^2$$

about the vertical axis. Since the differential of arc on the circle is

$$ds = \sqrt{1 + \left(\frac{dx}{dz}\right)^2}\, dz = \frac{a}{x}\, dz,$$

the area of the strip is

$$dS = 2\pi x\, ds = 2\pi a\, dz.$$

The center of gravity is on the axis of the shell at distance

$$\bar{z} = \frac{\int z\, dS}{\int dS} = \frac{\int_0^a 2\pi az\, dz}{\int_0^a 2\pi a\, dz} = \frac{a}{2}$$

from the center.

137. Theorems of Pappus. THEOREM I. *If the arc of a plane curve is revolved about an axis in its plane and not crossing the arc, the area generated is equal to the product of the length of the arc and the length of the path described by its centroid.*

Let the arc be rotated about the x-axis. The ordinate of the centroid is

$$\bar{y} = \frac{\int y\, ds}{s},$$

where s is the length of the arc. Thus

$$2\pi\bar{y}s = \int 2\pi y\, ds.$$

The right side of this equation represents the area generated, and $2\pi\bar{y}$ is the length of the path described by the centroid. This equation therefore expresses the result to be proved.

THEOREM II. *If a plane area is revolved about an axis in its plane and not crossing the area, the volume generated is equal to the product of the area and the length of path described by its centroid.*

Let the area A be revolved about the x-axis. Cut it into strips dA by lines parallel to the x-axis. The ordinate of the centroid is

$$\bar{y} = \frac{\int y\, dA}{A},$$

whence

$$2\pi\bar{y}A = \int 2\pi y\, dA.$$

The right side of this equation represents the volume generated, and $2\pi\bar{y}$ is the length of path described by the centroid. The equation therefore expresses the result to be proved.

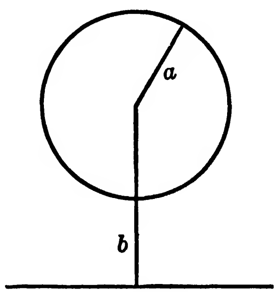

FIGURE 185.

Example 1. Find the area of the torus generated by revolving a circle of radius a about an axis in its plane at distance b (greater than a) from its center.

Since the circumference of the circle is $2\pi a$ and the length of the path described by its center $2\pi b$, the area of the torus is

$$S = 2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

Example 2. Use the theorems of Pappus to determine the centroid of a semicircular area.

When a semicircle of radius a is revolved about its diameter, it generates a sphere of volume $\frac{4}{3}\pi a^3$. If $\bar{y}$ is the distance from the centroid of the semicircle to this diameter, and $A = \frac{1}{2}\pi a^2$ is its area, by the second theorem of Pappus

$$\tfrac{4}{3}\pi a^3 = 2\pi\bar{y}A = \pi^2\bar{y}a^2,$$

whence

$$\bar{y} = \frac{\frac{4}{3}\pi a^3}{\pi^2 a^2} = \frac{4a}{3\pi}.$$

Example 3. Find the volume generated by revolving the cardioid

$$r = a(1 + \cos\theta)$$

about the initial line.

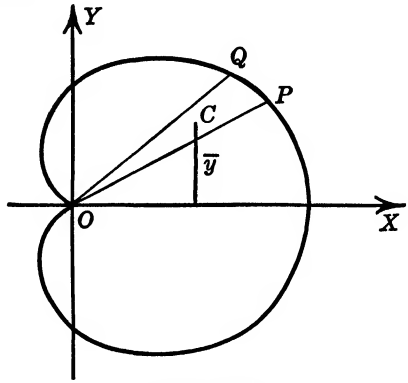

FIGURE 186.

The polar element of area OPQ (Figure 186) is approximately a triangle with area

$$\tfrac{1}{2}r^2\,d\theta$$

and centroid at distance

$$\bar{y} = \tfrac{2}{3}r\sin\theta$$

from the initial line. By the second theorem of Pappus the volume generated by OPQ is then approximately

$$2\pi\bar{y}\cdot\tfrac{1}{2}r^2\,d\theta = \tfrac{2}{3}\pi r^3\sin\theta\,d\theta.$$

The entire volume generated is therefore

$$v = \int_0^\pi \tfrac{2}{3}\pi r^3\sin\theta\,d\theta = \tfrac{2}{3}\pi a^3\int_0^\pi (1+\cos\theta)^3\sin\theta\,d\theta$$
$$= \left[-\frac{2}{3}\pi a^3\frac{(1+\cos\theta)^4}{4}\right]_0^\pi = \frac{8}{3}\pi a^3.$$

138. Moment of Inertia. The moment of inertia of a particle about an axis is the product of its mass and the square of its distance from the axis.

To find the moment of inertia of a continuous mass, we divide it into parts such that the points of each differ only infinitesimally in distance from the axis. Let Δm be the mass of such a part and r the distance of one of its points from the axis. The moment of inertia of this part is then approximately $r^2\,\Delta m$, and the moment of inertia of the entire mass is

$$I = \lim_{\Delta m \to 0} \Sigma r^2\,\Delta m = \int r^2\,dm. \tag{1}$$

If the mass of a body is M and its moment of inertia about a given axis is

$$I = Mk^2 \tag{2}$$

the distance k is called the *radius of gyration* of the body about that axis. It is the distance from the axis at which the body could be concentrated without changing its moment of inertia about that axis.

As an illustration of these formulas we take the moment of inertia of a homogeneous cylinder about its axis. To determine this cut the cylinder into pipe-shaped sections (Figure 187). Let a be the radius, h the altitude, and ρ the density. The section with inner radius r and outer radius $r + dr$ has the volume

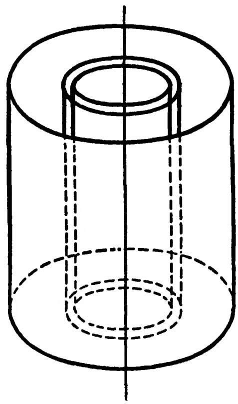

FIGURE 187.

$$dv = 2\pi rh\,dr$$

and mass

$$dm = \rho\,dv.$$

The moment of inertia of the cylinder is therefore

$$I = \int r^2\,dm = \int_0^a 2\pi\rho h r^3\,dr = \tfrac{1}{2}\pi\rho h a^4,$$

and its mass is

$$M = \pi\rho a^2 h.$$

Thus

$$I = \tfrac{1}{2}Ma^2 \tag{3}$$

is the moment of inertia of a homogeneous cylinder of mass M and radius a about its axis. Comparison with (2) shows that the radius of gyration in this case is

$$k = \frac{a}{\sqrt{2}}.$$

In certain discussions quantities appear that are similar to moments of inertia but with mass replaced by length, area, or volume. Such quantities are usually called moments of inertia although they may have no connection with inertia. Thus we have moments of inertia of length, area, or volume about an axis, these being represented by integrals of the form (1) but with dm replaced by an element of length, area, or volume.

Example 1. Find the moment of inertia of the area of a circle about a diameter of the circle.

Let the radius of the circle be a, and take the x-axis along the diameter about which the moment of inertia is taken. Divide the area into strips

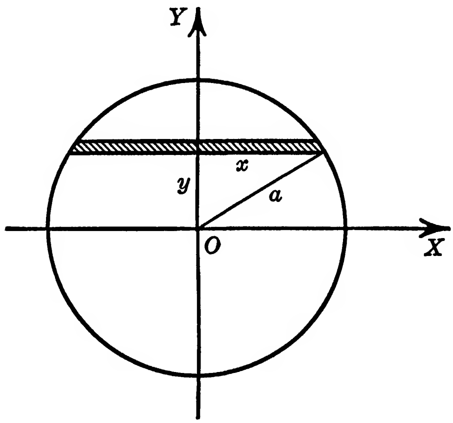

FIGURE 188.

by lines parallel to the x-axis. Neglecting infinitesimals of higher order than Δy, the area of each strip is $2x\,\Delta y$, and its moment of inertia is $2xy^2\,\Delta y$. The moment of inertia of the entire area is therefore

$$I = \int 2xy^2\,dy = 2\int_{-a}^{a} y^2\sqrt{a^2 - y^2}\,dy = \frac{\pi a^4}{4}.$$

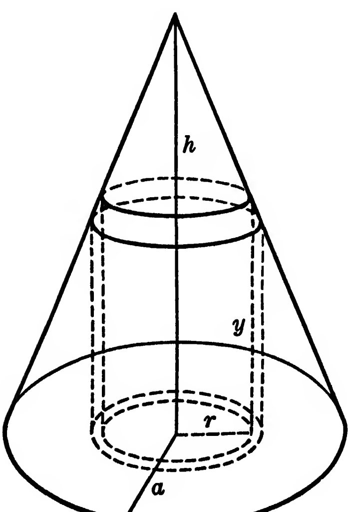

FIGURE 189.

Example 2. Find the moment of inertia of a right circular cone of constant density about its axis.

Let ρ be the density of the cone, h its altitude, and a the radius of its base. Divide the cone into hollow cylindrical (pipe-shaped) slices by cylindrical surfaces having the same axis as the cone. By similar triangles the altitude of the cylindrical surface of radius r is

$$y = \frac{h}{a}(a - r).$$

Neglecting infinitesimals of higher order than Δr, the volume between the surfaces of radii r, $r + \Delta r$ is

$$\Delta v = 2\pi r y\,\Delta r = \frac{2\pi h}{a} r(a - r)\,\Delta r,$$

the mass of this slice is $\rho\,\Delta v$, and its moment of inertia about the axis is $\rho r^2\,\Delta v$. The moment of inertia of the entire cone is therefore

$$I = \lim_{\Delta r \to 0} \Sigma \rho r^2\,\Delta v = \frac{2\pi h\rho}{a}\int_0^a r^3(a - r)\,dr = \frac{\pi\rho h a^4}{10}.$$

The mass of the cone is

$$M = \rho v = \tfrac{1}{3}\pi\rho a^2 h.$$

Thus

$$I = \tfrac{3}{10}Ma^2.$$

Example 3. The area bounded by the parabola

$$y^2 = 4x,$$

the x-axis, and the line $x = 4$ is rotated about the x-axis. Find the moment of inertia about the x-axis of the volume generated.

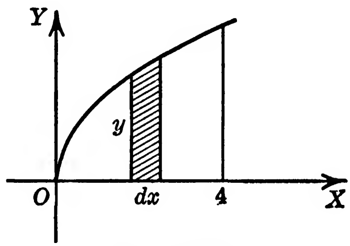

FIGURE 190.

The element of area $y\,dx$ generates a cylinder of radius y and volume

$$dv = \pi y^2\,dx.$$

By equation (3) its moment of inertia about the x-axis is

$$\frac{1}{2}y^2\,dv = \frac{\pi}{2}y^4\,dx = 8\pi x^2\,dx.$$

The moment of inertia of the entire volume is therefore

$$\int_0^4 8\pi x^2\,dx = [\tfrac{8}{3}\pi x^3]_0^4 = \tfrac{512}{3}\pi.$$

PROBLEMS

1. Find by integration the lateral area of a right circular cone of altitude h and radius of base a.

2. Find the area of the paraboloid of revolution generated by rotating about the x-axis the part of the parabola $y^2 = 4px$ between $x = 0$ and $x = p$.

3. Find the area generated by rotating about the y-axis the part of the parabola $x^2 = 3y$ between $x = 0$ and $x = 2$.

4. Find the area of a zone on a sphere of radius a between two parallel planes at distance h apart.

5. Find the area of the surface generated by rotating about the y-axis the part of the curve

$$8y = 9 - x^2$$

above the x-axis.

6. The arc of the curve

$$x^3 = 4y$$

between $x = 0$ and $x = 1$ is rotated about the x-axis. Find the area of the surface generated.

7. Find the area of the surface generated by rotating about the y-axis the arc of the tractrix

$$y = \frac{a}{2} \ln \frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}$$

between the points (x_1, y_1), (x_2, y_2).

8. Find the area of the sufrace generated by rotating the cardioid

$$r = a(1 + \cos \theta)$$

about the initial line.

9. Find the area of the surface generated by rotating the circle $r = a \sin \theta$ about the initial line.

10. Find the area of the surface generated by rotating the lemniscate

$$r^2 = 2a^2 \cos 2\theta$$

about the initial line.

11. Find the area of the surface generated by rotating the lemniscate $r^2 = 2a^2 \cos 2\theta$ about the line $\theta = \frac{\pi}{2}$.

12. Find the area of the surface generated by rotating the hypocycloid

$$x = a \cos^3 \phi, \quad y = a \sin^3 \phi$$

about the x-axis.

13. Find the area of the surface generated by rotating one arch of the cycloid

$$x = a(\phi - \sin \phi), \quad y = a(1 - \cos \phi)$$

about the x-axis.

14. The arc of the hyperbola

$$x = a \sec \phi, \quad y = a \tan \phi$$

between $\phi = 0$ and $\phi = \frac{\pi}{4}$ is rotated about the y-axis. Find the area of the surface generated.

15. The arc of the curve

$$x = a(\cos \phi + \phi \sin \phi), \quad y = a(\sin \phi - \phi \cos \phi)$$

between $\phi = 0$ and $\phi = \pi$ is rotated about the x-axis. Find the area generated.

16. At points of the curve $y^2 = 4px$, lines perpendicular to its plane of length $h = y$ are drawn. Find the area of the surface formed by these lines at points of the curve between $(0, 0)$ and $(p, 2p)$.

17. At each point of the circle $r = a \cos \theta$ a perpendicular to its plane is erected of length equal to r. Find the area of the cylindrical surface formed.

18. At points of a circle of radius a perpendiculars to its plane are erected, the perpendicular at each point P being of length ks, where s is the arc of the circle from a fixed point A to P. Find the area of the surface formed by the perpendiculars along the arc beginning at A and extending once around the circle.

19. The axes of two right circular cylinders of radius a intersect at right angles (Figure 52, page 90). Find the area of the solid common to the two cylinders.

20. A wedge is cut from a right circular cylinder of radius a by a plane inclined 45° to the base. Find the area of the curved surface of the wedge if the circular arc that forms its lower edge is an arc of 120°.

21. The angle between the axis of a cone and its generators is 45°. If the vertex of the cone is on the base of a cylinder of radius a and its axis is a generator of the cylinder find the area of the cylindrical surface below the cone.

22. One diameter of a circular cylinder of radius a is a radius of a sphere. Find the area of the cylindrical surface inside the sphere.

23. Find the mean value of $y = \sin x$ with respect to x between $x = 0$ and $x = \pi$.

24. Find the mean value of $\sin^2 x$ from $x = 0$ to $x = 2\pi$, the average being taken with respect to x.

25. Find the average value of $\sin 2x \sin 3x$ for a common period of $\sin 2x$ and $\sin 3x$, the average being taken with respect to x.

26. At points of a semicircle of radius a perpendiculars are drawn to the diameter through its ends. Find the mean value of these perpendiculars (1) if they are taken at equal intervals along the diameter, (2) if they are taken at equal intervals along the semicircular arc.

27. A body falls from rest under the acceleration of gravity. Find its mean velocity during the first t seconds if the average is taken with respect to the time. Show that the distance it falls in t seconds is this mean velocity multiplied by t.

28. A body falls from rest. Find its mean velocity during the first t seconds if the average is taken with respect to the distance.

29. From a point on a circle of radius a, lines are drawn to points equally spaced along the circumference. Find the mean of the squares of these lines.

30. Plane sections of a sphere are taken perpendicular to a diameter at points equally spaced along the diameter. Find the average area of section. Show that the volume of the sphere is this average area multiplied by the diameter.

31. A particle moves along a straight line in simple harmonic motion

$$x = a \sin kt.$$

Find the mean square of its velocity, the average being taken with respect to x.

32. An electric current of I amperes flowing through a resistance of R ohms produces heat at the rate of RI^2 watts. Find the average rate at which heat is produced by an alternating current

$$I = A \sin \omega t$$

flowing through a resistance of R ohms during one cycle $\left(t = 0 \text{ to } t = \dfrac{2\pi}{\omega}\right)$. Find the steady current I which produces the same total heat during one cycle.

33. A cubic foot of air is compressed from atmospheric pressure (14.7 pounds per square inch) to one-tenth its initial volume. If the gas satisfies Boyle's law $pv = k$, find the mean pressure, the average being taken with respect to the volume. Show that the work required to compress the gas is this mean pressure multiplied by the change in volume.

34. The wind produces a pressure of p pounds per square foot on a door b feet wide and h feet high. Find the torque tending to turn the door on its hinges.

35. Find the moment due to water pressure on a rectangular flood gate of width b and height h about a horizontal line through its center when the water level is at the top of the gate.

36. A dam has the form of a trapezoid with base 100 feet, width at top 400 feet, and height 30 feet. Assuming that the water is level with the top of the dam, find the moment of water pressure about the base line.

37. The diameter of a horizontal water main is 6 feet. One end is closed by a bulkhead and the other is connected with a reservoir in which the water surface is 30 feet above the center of the main. Find the moment of water pressure on the bulkhead about its horizontal diameter.

38. A triangular plate of constant thickness and density has a base b and altitude h. Find the distance of its center of gravity from the base. Take the element of area parallel to the base.

39. A trapezoid has two parallel sides of lengths 10 feet and 20 feet, and their distance apart is 12 feet. Find the distance from the 20-foot side to the centroid of its area.

40. Find the centroid of the area bounded by the parabola $y^2 = 4x$ and the line $x = 4$.

41. Find the centroid of the area above the x-axis bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

42. Find the centroid of the area bounded by the x-axis and the arc of the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

43. Find the centroid of the area bounded by the coördinate axes and the parabola

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$$

44. The two arms of a steel square are 2 inches and $1\frac{1}{2}$ inches wide. If the outer edges are 24 inches and 12 inches long, find its center of gravity.

45. From a semicircle of radius b a semicircle with the same center and radius a is cut. Find the centroid of the area left.

46. Find the centroid of the area bounded by the parabolas

$$y^2 = ax, \quad x^2 = ay.$$

Take an element of area parallel to a coördinate axis and extending between the parabolas.

47. Find the centroid of the area bounded by $y = 2x - x^2$ and $y = x$.

48. Find the centroid of the area on the right of the y-axis bounded by the curves

$$y = x^3 - 4x, \quad y = 2x - x^2.$$

49. Find the centroid of the arc of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

above the x-axis.

50. Find the centroid of the arc of the curve

$$x = \cos t, \quad y = \sin t$$

between $t = 0$ and $t = \frac{1}{3}\pi$.

51. Find the centroid of the area between the x-axis and one arch of the cycloid

$$x = a(\phi - \sin \phi), \quad y = a(1 - \cos \phi).$$

52. Find the center of gravity of a thin wire bent into the form of the arch of the cycloid in the preceding problem.

53. Find the centroid of the area within the cardioid

$$r = a(1 + \sin \theta).$$

54. Find the center of gravity of a thin wire bent into the form of the cardioid in the preceding problem.

55. A circular sector has radius a and central angle 2α. Find the centroid of its area.

56. A wire is bent into an arc of a circle of radius a and central angle 2α. Find its center of gravity.

57. Find the centroid of the area within the curve

$$r = a(2 - \cos \theta).$$

58. Find the centroid of the area bounded by the parabola

$$r = a \sec^2 \frac{\theta}{2}$$

and the lines $\theta = \pm\dfrac{\pi}{3}$.

59. Find the centroid of the area within one loop of the lemniscate

$$r^2 = 2a^2 \cos 2\theta.$$

60. Find the center of gravity of a right circular cone of altitude h and constant density.

61. Find the center of gravity of a thin shell of constant thickness covering the surface of a right circular cone of altitude h.

62. The area bounded by the curve $y^2 = ax$ and the line $x = a$ is rotated about the x-axis. Find the centroid of the volume generated.

63. Find the centroid of the volume generated when the area above the x-axis in the preceding problem is rotated about the y-axis.

64. The area in the first quadrant bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is rotated about the y-axis. Find the centroid of the volume generated.

65. The area above the x-axis bounded by the circle $x^2 + y^2 = a^2$ and the straight line $x + y = a$ is rotated about the x-axis. Find the centroid of the resulting volume.

66. A hemispherical shell of constant density has the inner radius a and outer radius b. Find its center of gravity. Consider it as the difference of two hemispheres (compare example 1, §136).

67. By finding the limit as b approaches a in the preceding problem find the centroid of a hemispherical surface.

68. If Boyle's law holds, the density of air at height h is

$$\rho = \rho_0 e^{-\frac{h}{k}},$$

ρ_0 and k being positive constants. Assuming this, find the center of gravity of a vertical cylinder of air reaching from the surface of the earth to infinity.

69. Under the adiabatic law the density of air at height h is

$$\rho = a(b - h)^m,$$

a, b, m being positive constants. Assuming this, find the center of gravity of a vertical cylinder of air reaching from the earth's surface to the top of the atmosphere.

70. The center of pressure on a submerged plane area is defined as the point at which the total force could be concentrated without changing its total moment about any axis in the plane. Find the center of pressure on a vertical rectangle of height h and width b with its upper edge in the surface of the water.

71. Find the center of pressure on a triangle of base b and altitude h if its base is in the surface of the water and its altitude is vertical.

72. Show that the force due to liquid pressure on a submerged plane area is equal to the product of the area by the pressure at the center of gravity. As an application determine the force on a triangle of base b and altitude h with its vertex in the surface and altitude vertical.

73. By using the theorems of Pappus find the lateral area and the volume of a right circular cone.

74. Find the volume generated by rotating a circle of radius a about an axis at distance b (greater than a) from its center.

75. By the theorems of Pappus find the centroid of a semicircular arc.

76. A groove with cross section an equilateral triangle of side 1 inch is cut around a shaft 6 inches in diameter. Find the volume of metal cut away.

77. A groove with cross section a semicircle of diameter 1 inch is cut around a shaft 6 inches in diameter. Find the volume of metal cut away (cf. example 2, §137).

78. The length of an arch of the cycloid

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi)$$

is $8a$, and the area generated by rotating it about the x-axis is $\dfrac{64\pi a^2}{3}$. Determine the centroid of the arch, and then find the area generated by rotating it about the tangent at its highest point.

79. Find the area generated by rotating the arc in the preceding problem about the y-axis.

80. A circular sector is bounded by the circle $r = a$ and the radii $\theta = \alpha$, $\theta = \beta$ $(\alpha > \beta > 0)$. By the method of example 3, §137, find the volume generated by rotating this sector about the initial line.

81. Find the centroid of the volume in the preceding problem. Use as elements of volume the hollow cones generated by rotating the polar elements of area and the fact that the centroid of a cone is three-fourths of the way from its vertex to its base (cf. problem 60).

82. Find the volume generated by rotating about the initial line the area within the curve $r = a(2 - \cos\theta)$.

83. Find the volume generated by rotating the area within the lemniscate

$$r^2 = 2a^2 \cos 2\theta$$

about the initial line.

84. The sides of a rectangle are a and b. Find the moment of inertia of its area about the side of length a.

85. Find the moment of inertia of the area of a triangle of base b and altitude h about the axis through its vertex parallel to its base.

86. A thin circular plate has radius a and mass M. Find its moment of inertia about a diameter.

87. Find the moment of inertia of the area within the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

88. Find the moment of inertia of a uniform straight wire of length l and mass M about an axis perpendicular to the wire at one end.

89. A uniform wire of mass M is bent into a circle of radius a. Find its moment of inertia about a diameter of the circle.

90. The rim of a flywheel has inner and outer radius r_1, r_2, and its mass is M. Find its moment of inertia about its axis.

91. The area within the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is rotated about the x-axis. Find the moment of inertia of the resulting volume about the x-axis.

92. The area bounded by the x-axis, the curve $y = x^2$ and the line $x = 2$ is rotated about the x-axis. Find the moment of inertia of the resulting volume about the x-axis.

93. A thin conical shell of constant thickness has the altitude h, radius of base a, and mass M. Find its moment of inertia about its axis.

94. Find the radius of gyration of a uniform spherical ball of radius a about a diameter.

95. Find the radius of gyration of a thin spherical shell of radius a and constant thickness and density about a diameter.

96. A ring is generated by rotating a circle of radius a about an axis at distance $b > a$ from its center. Find the radius of gyration of the ring.

97. The kinetic energy of a moving mass is

$$\int \tfrac{1}{2} v^2 \, dm,$$

where v is the velocity of the element of mass dm. Find the kinetic energy of a homogeneous cylinder of radius a and mass M rotating with angular velocity ω about its axis.

98. Find the kinetic energy of a sphere of radius a and mass M rotating with angular velocity ω about a diameter.

CHAPTER XIII

SERIES WITH REAL TERMS

139. Convergence. If $u_1, u_2, \cdots, u_n, \cdots$ is a given infinite sequence of numbers, the expression

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots, \tag{1}$$

often abbreviated Σu_n, is called an infinite series and u_1, u_2, u_3, etc., are called its *terms*. For the present we assume all the terms real.

If the sum

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n \tag{2}$$

of the first n terms tends to a limit as $n \to \infty$, the series is said to converge and the limit

$$S = \lim_{n \to \infty} S_n \tag{3}$$

is called its *sum*. We use the notation

$$S = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

to indicate that the series converges and that S is the sum.

Thus, if $r \neq 1$, the first n terms of the series

$$a + ar + ar^2 + \cdots + ar^n + \cdots \tag{4}$$

have for sum

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r}. \tag{5}$$

If $|r| < 1$ this tends to the limit

$$S = \frac{a}{1 - r} \tag{6}$$

as n increases. The series (4), called a *geometric series*, is thus convergent when $|r| < 1$ and has for sum the value S determined by (6).

A series that is not convergent is called *divergent.*

Suppose, for example, that the constant a in (4) is not zero. If $r = 1$ the series becomes

$$a + a + a + \cdots + a + \cdots$$

and the sum of n terms, $S_n = na$, obviously does not tend to a limit as n increases. If $r = -1$ the series takes the form

$$a - a + a - a \cdots \pm a \mp a \cdots,$$

and S_n has the value a if n is odd and 0 if n is even. In this case S_n also does not tend to a limit. Finally, if $|r| > 1$, equation (5) shows again that S_n does not tend to a limit as $n \to \infty$. The geometric series therefore diverges when a is not zero and $|r| \geqq 1$.

140. General Condition for Convergence. The condition that the partial sum S_n tends to a limit as n increases is merely that the differences $S_m - S_n$ become as small as you please when m and n are sufficiently large [§14, (1)]. Since

$$S_{n+p} - S_n = u_{n+1} + u_{n+2} + \cdots + u_{n+p}, \tag{1}$$

the general condition for convergence is that, however small the positive number ϵ may be, there is an integer N such that, if $n > N$ and p is any positive integer,

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon. \tag{2}$$

This amounts to saying the condition for convergence is that there be a place in the series beyond which the sum of any number whatever of *consecutive* terms is in absolute value less than a preassigned positive ϵ. In particular this requires that any single term beyond that point be in absolute value less than ϵ. Thus,

The terms u_n of a convergent series tend to zero as $n \to \infty$.

The converse of this is, however, not true. That is, the terms of a series may tend to zero although it does not converge. Thus, the terms of

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

tend to zero as n increases, but we shall presently see [§142, (7)] that this series does not converge.

141. Positive Series. The simplest series are those in which all terms have the same algebraic sign. In particular we call the series *positive* if each of its terms is positive or zero.

The sum S_n of the first n terms in a positive series increases, or at least does not decrease, as n increases. This sum therefore tends to a limit if it is bounded [§14, (2)]. As an alternative to the general convergence condition [§140, (2)] we thus have:

A necessary and sufficient condition for the convergence of a positive series is that there exist some number A such that

$$S_n < A$$

for all values of n.

Whether a positive series converges or not is usually determined by comparing it with an integral or with another series that is known to converge or diverge. In making these tests it should be noted that a finite number of terms in a series may be omitted without influencing convergence, provided these terms have finite values.

142. Integral Test. The nth term in a series is a function of n, say

$$u_n = f(n). \tag{1}$$

The series can thus be written

$$f(1) + f(2) + f(3) + \cdots + f(n) + \cdots. \tag{2}$$

This function $f(n)$ is usually defined not only for integral values but for all values of n. With the series we can therefore associate an integral

$$\int_1^\infty f(x)\,dx. \tag{3}$$

In many cases the convergence of the series can be deduced from that of the integral, the basis for this deduction being the following theorem:

If $f(x)$ is positive and always decreases as x increases, the series (2) *and the integral* (3) *both converge or both diverge.*

To prove this we note that since $f(x)$ is a decreasing function

$$\int_n^{n+1} [f(n) - f(x)]\,dx > 0, \quad \int_n^{n+1} [f(x) - f(n+1)]\,dx > 0,$$

whence

$$f(n) > \int_n^{n+1} f(x)\,dx > f(n+1). \tag{4}$$

Writing this inequality for $n = 1, 2, 3, \cdots, p$, and adding, we get

$$f(1) + f(2) + \cdots + f(p) > \int_1^{p+1} f(x)\,dx$$
$$> f(2) + f(3) + \cdots + f(p + 1). \quad (5)$$

Since $f(x)$ is positive, the integral

$$\int_1^{p+1} f(x)\,dx$$

and the sum

$$f(1) + f(2) + \cdots + f(p)$$

both increase as p increases and so both tend to limits if they are bounded [§14, (2)]. The left side of (5) shows that the integral is bounded if the sum is, and the right side shows that the sum is bounded if the integral is. Thus the series and the integral both converge or both diverge.

As an illustration take the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, \quad (6)$$

where p is any positive number. The nth term

$$f(n) = \frac{1}{n^p}$$

is positive and decreases as n increases. The series is thus convergent if the integral

$$\int_1^\infty \frac{dx}{x^p}$$

is convergent, that is if $p > 1$, and divergent if the integral is divergent, that is if $p \leqq 1$. This discussion shows, in particular, that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \quad (7)$$

is divergent.

143. Comparison Test. When the convergence or divergence of a certain series is known that of others can often be inferred, the principle employed being the following:

The positive series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1}$$

is convergent if the positive series

$$b_1 + b_2 + b_3 + \cdots + b_n + \cdots \tag{2}$$

is convergent and either

$$a_n \leqq kb_n, \tag{3}$$

where k is constant, or

$$\frac{a_{n+1}}{a_n} \leqq \frac{b_{n+1}}{b_n} \tag{4}$$

for all values of n.

The positive series (1) *is divergent if the positive series* (2) *is divergent and either*

$$a_n \geqq kb_n, \tag{5}$$

where k is a constant greater than zero, or

$$\frac{a_{n+1}}{a_n} \geqq \frac{b_{n+1}}{b_n} \tag{6}$$

for all values of n.

The proof of these statements is immediate. Thus, if the condition (3) is satisfied,

$$a_1 + a_2 + \cdots + a_n \leqq k(b_1 + b_2 + \cdots + b_n)$$

and so the sum of n terms in the series (1) is bounded if the sum of n terms in the series (2) is bounded. The condition (4) can be written

$$\frac{a_{n+1}}{b_{n+1}} \leqq \frac{a_n}{b_n},$$

whence by induction

$$\frac{a_n}{b_n} \leqq \frac{a_1}{b_1}.$$

Thus (4) is equivalent to (3) with

$$k = \frac{a_1}{b_1}.$$

As we run through the series Σa_n the ratio

$$\frac{a_{n+1}}{a_n}$$

measures the rate of growth of the terms. The parts (4) and (6) of the above theorem thus state that a positive series is convergent if the rate of growth of its terms is ultimately less than that of a given convergent positive series, and is divergent if the rate of growth is ultimately greater than that of a given divergent positive series.

Example 1. The series

$$\frac{2}{1} + \frac{2}{2\cdot 3} + \frac{2}{3\cdot 5} + \cdots + \frac{2}{n(2n-1)} + \cdots$$

is convergent since the ratio of its nth term to that in the convergent series [§142, (6)]

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

is

$$\frac{2n}{2n-1} \leqq 2.$$

Example 2. The series

$$\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} + \cdots$$

is divergent since the ratio of consecutive terms

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2(n+1)} = \frac{n+\frac{1}{2}}{n+1}$$

is greater than the corresponding ratio

$$\frac{b_{n+1}}{b_n} = \frac{n}{n+1}$$

in the divergent series [§142, (7)]

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

144. Absolute and Conditional Convergence. In the series we have been discussing all the terms have been positive. If a series contains both positive and negative terms, it is necessary to distinguish two kinds of convergence.

The series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{1}$$

is called *absolutely convergent* if the series of absolute values

$$|u_1| + |u_2| + |u_3| + \cdots + |u_n| + \cdots \tag{2}$$

is convergent. A series that converges, but is not absolutely convergent, is called *conditionally convergent.*

It should be noted that *an absolutely convergent series is convergent.* By that we mean that the series (1) is convergent if (2) is convergent. This follows since

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| \leqq |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}|,$$

and so the general convergence condition [§140, (2)] is satisfied by the terms of (1) if it is satisfied by those of (2).

145. Ratio Test. The following, called the ratio test, is particularly useful when each term is formed from the preceding term by a comparatively simple multiplication:

Let

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{1}$$

be an infinite series and

$$L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|. \tag{2}$$

If $L < 1$, the series is absolutely convergent; if $L > 1$, the series is divergent; if $L = 1$, the series may converge or may diverge.

First suppose that $L < 1$ and let r be any number that satisfies the condition

$$L < r < 1. \tag{3}$$

As n increases,

$$\left| \frac{u_{n+1}}{u_n} \right|$$

ultimately becomes as nearly equal to L as we please and so ultimately remains between L and r. Disregarding a finite number of terms, which has no influence on convergence, we thus have

$$\left| \frac{u_{n+1}}{u_n} \right| < r.$$

Since $r < 1$, the geometric series

$$1 + r + r^2 + \cdots + r^n + \cdots$$

is convergent [§139, (4)]. The series

$$|u_1| + |u_2| + \cdots + |u_n| + \cdots$$

then also converges since the ratio of consecutive terms

$$\left| \frac{u_{n+1}}{u_n} \right|$$

is less than the corresponding ratio

$$\frac{r^{n+1}}{r^n} = r$$

in the geometric series [§143, (4)].

If $L > 1$, the series (1) obviously diverges since the terms u_n do not then tend to zero as n increases. If $L = 1$, examples show that the series may converge or may diverge.

Example 1. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$

This is the series used to define the number e [§84, (3)]. In this case

$$L = \lim_{n \to \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \to \infty} \frac{1}{n+1} = 0.$$

Since $L < 1$, the series is convergent.

Example 2. $1 + 2r + 3r^2 + \cdots + (n + 1)r^n + \cdots.$

In this case

$$L = \lim_{n \to \infty} \frac{(n+2)\,|r|}{n+1} = |r|.$$

The series thus converges if $|r| < 1$ and diverges if $|r| > 1$. The series also diverges when $|r| = 1$, for the nth term does not tend to zero as n increases.

146. Alternating Series. A series with terms alternately positive and negative is called an *alternating series.* The convergence of such a series and the error due to using n terms as an approximation can often be determined by the following theorem:

A series

$$a_1 - a_2 + a_3 - a_4 + \cdots, \tag{1}$$

with terms alternately positive and negative, is convergent if each term is numerically less than the preceding and

$$\lim_{n \to \infty} a_n = 0. \tag{2}$$

The error committed by stopping at the nth term is less than the first term omitted.

To prove this we note that the sum of $2m$ terms can be written in the two forms

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}) \tag{3}$$

$$= a_1 - (a_2 - a_3) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}. \tag{4}$$

Since a_n diminishes as n increases, each expression in parentheses is positive. Equation (3) thus shows that S_{2m} increases as m increases and (4) that it always remains less than a_1. Therefore S_{2m} approaches a limit as m increases [§14, (2)]. Since

$$S_{2m+1} = S_{2m} + a_{2m+1}$$

and a_{2m+1} tends to zero, S_{2m+1} tends to the same limit. Thus, whether n varies through even or odd values, S_n approaches a definite limit as n increases. The series is therefore convergent. If we stop with the nth term, the error

$$a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots$$
$$= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) \cdots$$

is obviously less than a_{n+1}.

Example. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

Since the terms are alternately positive and negative and steadily diminish to the limit zero, the series is convergent. Since

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent, the given series is, however, not absolutely, but only conditionally, convergent. If

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4}$$

is used as an approximation for the sum of the series, the error will be less than $\frac{1}{5}$.

147. Series of Functions. In the series we have hitherto discussed, the terms have been constant. If the terms are functions of a variable x, the series has the form

$$u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots. \tag{1}$$

This is said to converge for a particular value $x = c$ if the series of constants

$$u_1(c) + u_2(c) + u_3(c) + \cdots + u_n(c) + \cdots$$

is convergent. The set of values x for which a series converges constitute its *region of convergence.*

For values in its region of convergence the sum of the series $\Sigma u_n(x)$ is a function

$$S(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots. \tag{2}$$

This can be expressed as the sum of two parts

$$S(x) = S_n(x) + R_n(x),$$

where

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) \tag{3}$$

is the sum of the first n terms and

$$R_n(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x) + \cdots \tag{4}$$

is the *remainder* after n terms. At a point where the series converges $S_n(x)$ tends to $S(x)$ as n increases and $R_n(x)$ tends to zero.

148. Power Series. A series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots, \tag{1}$$

where a_0, a_1, a_2, etc., are constants, and x variable, is called a power series. More generally

$$a_0 + a_1(x - a) + \cdots + a_n(x - a)^n + \cdots, \tag{2}$$

where a is constant, is also a power series which differs from (1) only in the origin from which x is measured. To distinguish the two forms we sometimes call (1) a power series in x and (2) a power series in $x - a$.

With regard to convergence of power series the following is fundamental:

If a power series in x converges when $x = b$, it converges absolutely for every value of x numerically less than b.

In fact, if the series

$$a_0 + a_1b + a_2b^2 + \cdots + a_nb^n + \cdots \tag{3}$$

converges, the terms a_nb^n tend to zero as n increases (§140). They are therefore all numerically less than some number M, that is

$$|a_nb^n| < M, \quad |a_n| < \frac{M}{|b|^n}. \tag{4}$$

Thus the terms of the series

$$|a_0| + |a_1x| + |a_2x^2| + \cdots + |a_nx^n| + \cdots \tag{5}$$

are respectively less than those of the geometric series

$$M + \frac{M}{|b|}|x| + \frac{M}{|b|^2}|x|^2 + \cdots + \frac{M}{|b|^n}|x|^n + \cdots, \tag{6}$$

in which the ratio of consecutive terms is

$$r = \frac{|x|}{|b|}. \tag{7}$$

If then $|x| < |b|$, the geometric series, and consequently (5), converges, that is, (1) converges absolutely.

If a power series in x diverges when $x = b$, it diverges for every value of x numerically greater than b.

For it could not converge when $|x| > |b|$ since, by the proof just given, it would then converge at $x = b$.

A power series in x obviously converges when $x = 0$. The above theorems show that it either converges for all values of x or there is a numerical value beyond which it everywhere diverges. If we consider $x = 0$ as a zero interval and the totality of values an infinite one, we can then say that *the region of convergence of a power series in x is an interval of center $x = 0$, and inside that interval the series converges absolutely.*

The limits of the convergence interval are usually determined by the ratio test (§145). At the ends of the interval the series may converge or may diverge, or it may converge at one end and diverge at the other.

A zero convergence interval is illustrated by the series

$$1 + x + (2!)x^2 + (3!)x^3 + \cdots + (n!)x^n + \cdots.$$

The ratio of consecutive terms in this series is

$$\frac{u_{n+1}}{u_n} = (n+1)x,$$

and unless $x = 0$ this is numerically greater than 1 when n is sufficiently large. Thus the series diverges for every value except $x = 0$ (§145).

A finite convergence interval is illustrated by the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

which we have already seen (§139) converges absolutely within the interval

$$-1 < x < 1$$

and diverges at the ends of the interval and outside.

As a series with infinite interval of convergence take

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

In this case the ratio of consecutive terms is

$$\frac{u_{n+1}}{u_n} = \frac{x}{n+1}.$$

For any fixed value of x this has the limit

$$L = \lim_{n \to \infty} \frac{x}{n+1} = 0.$$

Thus the series converges absolutely for all values of x (§145).

Series in powers of $x - a$ are discussed in a quite similar way, the essential difference being that the center of the convergence interval is $x = a$ instead of $x = 0$.

Example. $1 + \dfrac{x-2}{3} + \dfrac{(x-2)^2}{3^2} + \cdots + \dfrac{(x-2)^n}{3^n} + \cdots.$

The ratio of consecutive terms is

$$\frac{u_{n+1}}{u_n} = \frac{x-2}{3}.$$

The series therefore converges if

$$L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{|x-2|}{3} < 1,$$

that is, if

$$|x - 2| < 3.$$

The values of x that satisfy this condition extend the distance 3 on each side of $x = 2$ and therefore form the interval

$$-1 < x < 5.$$

At the limiting values $x = -1$ and $x = 5$ the series is divergent since its nth term does not then tend to zero as n increases.

149. Taylor's and Maclaurin's Series. For values of x in its region of convergence a power series in x or $x - a$ represents a function of x. Conversely, most of the commonly used functions can be represented as power series in $x - a$, at least for certain values of x and a. Assuming that a given function can be so represented, the coefficients in the series are readily determined by differentiation.

Thus suppose the function $f(x)$ expressible as a power series of the form

$$f(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots, \quad (1)$$

the representation being valid within the convergence interval of the series. Differentiating both sides successively with respect to x and assuming that the derivatives can be obtained by differentiating the series term by term, we find

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x - a) + 3a_3(x - a)^2 + 4a_4(x - a)^3 + \cdots, \\ f''(x) &= 2a_2 + 2\cdot 3a_3(x - a) + 3\cdot 4a_4(x - a)^2 + \cdots, \\ f'''(x) &= 1\cdot 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4(x - a) + \cdots, \end{aligned}$$

etc. For values of x in a sufficiently small interval of center $x = a$, these equations are assumed to be identities. Putting $x = a$, they give

$$f(a) = a_0, \quad f'(a) = a_1, \quad f''(a) = (2!)a_2, \quad \cdots.$$

Solving for a_0, a_1, a_2, etc., and substituting in (1), we finally obtain

$$\begin{aligned} f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} &+ \cdots \\ + f^n(a)\frac{(x - a)^n}{n!} &+ \cdots. \quad (2) \end{aligned}$$

This is known as *Taylor's series*. It is valid in a given interval of center $x = a$, provided that $f(x)$ is expressible as a power series convergent in that interval. The final result will be investigated later in another way (§150).

When $a = 0$, Taylor's series becomes

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^n(0)\frac{x^n}{n!} + \cdots, \tag{3}$$

which is called *Maclaurin's series.*

Taylor's series gives $f(x)$ in terms of the values that $f(x)$ and its derivatives assume at $x = a$. In using the series to compute $f(x)$, the constant a may to a certain extent be assigned arbitrarily. The value chosen must, however, be one for which $f(a)$, $f'(a)$, $f''(a)$, etc., are all known. Furthermore it should be taken as close as possible to the value x at which $f(x)$ is wanted. For, the smaller $x - a$, the fewer terms $(x - a)^2$, $(x - a)^3$, etc., need be computed to obtain a desired approximation. In particular Maclaurin's series should be used when x is small.

Since Taylor's series converges most rapidly near $x = a$, it is often called the expansion of $f(x)$ in the neighborhood of $x = a$. It is a series in powers of $x - a$, and the above discussion shows that for a given function $f(x)$ and value a there is only one such series.

Similarly, Maclaurin's series is the expansion of $f(x)$ in powers of x, or expansion in the neighborhood of $x = 0$.

Example 1. Expand $(1 + x)^m$ as a series in powers of x, and determine its region of convergence.

In this case

$$\begin{aligned} f(x) &= (1 + x)^m, & f(0) &= 1, \\ f'(x) &= m(1 + x)^{m-1}, & f'(0) &= m, \\ f''(x) &= m(m - 1)(1 + x)^{m-2}, & f''(0) &= m(m - 1), \end{aligned}$$

etc. Substituting these values, (3) becomes

$$\begin{aligned} (1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots \\ + \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots. \end{aligned} \tag{4}$$

This is called the *binomial series.* To determine its region of convergence we take the ratio of consecutive terms

$$\frac{u_{n+1}}{u_n} = \frac{m-n}{n+1}x$$

and find the limit

$$L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = |x|.$$

By the ratio test (§145) the series converges if $|x| < 1$ and diverges if $|x| > 1$.

Example 2. Expand $\ln x$ in powers of $x - 1$, obtaining four non-vanishing terms.

In this case, $a = 1$,

$$f(x) = \ln x, \qquad f(1) = 0,$$

$$f'(x) = \frac{1}{x}, \qquad f'(1) = 1,$$

$$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1,$$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2,$$

$$f^{\text{iv}}(x) = -\frac{6}{x^4}, \qquad f^{\text{iv}}(1) = -6,$$

etc. Substituting these values, (2) becomes

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots. \tag{5}$$

Example 3. Find $\sqrt[5]{33}$ correct to four figures.

The number nearest 33 with integral fifth root is 32. We therefore write

$$\sqrt[5]{33} = \sqrt[5]{32+1} = 2\sqrt[5]{1 + \tfrac{1}{32}}.$$

By the binomial series

$$\left(1 + \frac{1}{32}\right)^{\frac{1}{5}} = 1 + \frac{1}{5}\cdot\frac{1}{32} + \frac{\frac{1}{5}(-\frac{4}{5})}{2}\,\frac{1}{(32)^2} + \cdots$$

$$= 1 + \frac{1}{160} - \frac{1}{12{,}800} + \cdots.$$

Since this is an alternating series, the value

$$\sqrt[5]{33} = 2(1 + \tfrac{1}{160}) = 2.012$$

evidently has the required precision.

Example 4. Find $\tan(46°)$ to four decimals.

The value nearest to 46° at which $\tan x$ and its derivatives are known is 45°. We therefore let $a = \dfrac{\pi}{4}$.

$$f(x) = \tan x, \qquad f\left(\frac{\pi}{4}\right) = 1,$$

$$f'(x) = \sec^2 x, \qquad f'\left(\frac{\pi}{4}\right) = 2,$$

$$f''(x) = 2\sec^2 x \tan x, \qquad f''\left(\frac{\pi}{4}\right) = 4,$$

$$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x, \quad f'''\left(\frac{\pi}{4}\right) = 16.$$

Using these values, Taylor's series assumes the form

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \cdots.$$

When $x = 46°$

$$x - \frac{\pi}{4} = \frac{\pi}{180} = .01745.$$

Powers beyond the second are evidently too small to influence the fourth decimal. Hence

$$\tan 46° = 1 + 2(.01745) + 2(.01745)^2 = 1.0355.$$

150. Taylor's Series with Remainder. To obtain Taylor's series we assumed that $f(x)$ could be represented by a convergent series in powers of $x - a$. We shall now derive a formula that frequently can be used to justify that assumption. For that purpose let R_n be the function of x and a such that

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x-a)^2}{2!} + \cdots$$
$$+ f^n(a)\frac{(x-a)^n}{n!} + R_n. \quad (1)$$

This function R_n is called the *remainder* in Taylor's series. It is the error committed when we drop all powers of $x - a$ beyond the nth. If R_n tends to the limit zero as n increases, the series is convergent and its sum is $f(x)$.

To find an expression for R_n let $P_n(x, a)$ be a function of x and a defined by the equation

$$R_n = P_n(x, a)(x - a)^{n+1} \quad (2)$$

and

$$F(t) = f(x) - f(t) - f'(t)(x - t) - \cdots$$
$$- f^n(t)\frac{(x-t)^n}{n!} - P_n(x, a)(x - t)^{n+1}. \quad (3)$$

In this equation consider x and a as constants and t as variable. Equations (1), (2), and (3) show that $F(a) = 0$. It is also obvious that $F(x) = 0$. By the mean value theorem (§34), if $F(t)$ is dif-

ferentiable, that is, if $f(t)$ has derivatives up to the $(n + 1)$st order, there is then a value ξ between a and x such that

$$F'(\xi) = 0.$$

Differentiating $F(t)$ with respect to t, and replacing t by ξ, we get

$$F'(\xi) = -f^{n+1}(\xi)\frac{(x - \xi)^n}{n!} + (n + 1)P_n(x, a)(x - \xi)^n = 0$$

whence

$$P_n(x, a) = \frac{f^{n+1}(\xi)}{(n + 1)!}. \tag{4}$$

By substituting this in (2) we obtain the value R_n in (5) below. Thus we have proved:

If $f(t)$ is continuous in the interval $a \leqq t \leqq x$, and has a derivative of order $n + 1$ for each value of t between a and x, there is a value ξ between a and x such that the remainder in Taylor's series is

$$R_n = f^{n+1}(\xi)\frac{(x - a)^{n+1}}{(n + 1)!}. \tag{5}$$

This may be regarded as an extension of the mean value theorem. If we stop the series with the nth term, a limit to the possible error is obtained by assigning to ξ the value between a and x which makes $f^{n+1}(\xi)$ largest. Unless $|x - a|$ is very small this limiting error may, however, be considerably too large. A better estimate for the actual error is usually given by taking ξ midway between a and x.

Example 1. Prove that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \tag{6}$$

for all values of x.

By substituting in (1) with $a = 0$ we get

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n.$$

By equation (5)

$$R_n = \frac{e^{\xi}x^{n+1}}{(n + 1)!}$$

where ξ is between 0 and x. Thus

$$|R_n| \leqq \frac{e^{|x|}|x|^{n+1}}{(n + 1)!}.$$

For any fixed value of x this tends to zero as n increases. The series (6) therefore converges to e^x as limit for every value of x.

Example 2. Find the maximum error in the equation

$$\ln (1 + x) = x - \frac{x^2}{2}$$

if $| x | < 0.2$.

The remainder in the Maclaurin expansion is

$$R_2 = \frac{x^3}{3(1 + \xi)^3},$$

where ξ is between 0 and x. For $| x |$, and consequently $| \xi |$, less than 0.2, the greatest value of this expression is given by

$$x = \xi = -0.2.$$

Thus

$$| R_2 | \leqq \frac{(0.2)^3}{3(1 - 0.2)^3} = 0.0052.$$

If we take for ξ the middle value -0.1 instead of the extreme value, the maximum error thus estimated is

$$\frac{(0.2)^3}{3(1 - 0.1)^3} = 0.0037.$$

The actual maximum is

$$(-0.2) - \frac{(0.2)^2}{2} - \ln (0.8) = 0.0032.$$

151. Operations with Power Series. In books on advanced calculus it is shown that within its convergence interval (including an end point at which the series converges) the function

$$f(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots \tag{1}$$

represented by a power series is continuous and its integral

$$\int f(x)\, dx = c + a_0(x - a) + \frac{a_1}{2}(x - a)^2 + \cdots \tag{2}$$

may be obtained by integrating the series term by term. Within the convergence interval (but possibly not including the ends) the derivative

$$f'(x) = a_1 + 2a_2(x - a) + \cdots + na_n(x - a)^{n-1} + \cdots \tag{3}$$

is also obtained by differentiating term by term.

If two or more series of the form (1) converge in an interval of center $x = a$, they may be added, subtracted, or multiplied and

the resulting series in powers of $x - a$ will converge within that interval. Provided that the denominator does not vanish, one series of the form (1) can even be divided by another. The quotient arranged in powers of $x - a$ will converge at $x = a$, but its convergence interval may be smaller than that of either series.

Example 1. The series

$$\frac{1}{1 + t^2} = 1 - t^2 + t^4 - t^6 + \cdots \tag{4}$$

is convergent if $|t| < 1$. By integrating both sides we obtain

$$\tan^{-1} x = \int_0^x \frac{dt}{1 + t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \tag{5}$$

convergent if $|x| < 1$. This series converges at $x = 1$, and so by continuity we have

$$\tan^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{6}$$

although (4), which was used in its derivation, does not converge at $t = 1$.

Example 2. The series

$$\ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

converges in the interval

$$-1 < x \leqq 1.$$

By differentiating both sides we obtain

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots,$$

convergent if $|x| < 1$ but not at the limit $x = 1$.

Example 3. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,$$

converge for all values of x. By multiplication term by term, we obtain

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots$$

valid for all values of x.

Example 4. The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

converge for all values of x. By division we get

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots,$$

convergent in the interval

$$-\frac{\pi}{2} < x < \frac{\pi}{2}.$$

PROBLEMS

1. The sum of n terms of a series is

$$S_n = u_1 + u_2 + \cdots + u_n = \frac{n+1}{n}.$$

Show that the series converges, and find its sum.

2. The sum of n terms of a series is

$$S_n = u_1 + u_2 + \cdots + u_n = \frac{1}{n^2}.$$

Find the terms of the series and their sum.

3. Show that

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}.$$

By writing this equation for $m = 1, 2, \cdots, n$ and adding, find the sum of n terms of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots,$$

and show that the sum of the series is 1.

4. Show that

$$\frac{1}{(m+a)(m+1+a)} = \frac{1}{m+a} - \frac{1}{m+1+a}.$$

By means of this identity determine the sum of the series

$$\frac{1}{(1+a)(2+a)} + \frac{1}{(2+a)(3+a)} + \cdots + \frac{1}{(n+a)(n+1+a)} + \cdots.$$

By finding the sum of n terms determine whether the following series converge or diverge:

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^n.$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n} - \frac{\cos(n+1)}{n+1}.$

7. $\displaystyle\sum_{n=0}^{\infty} \cos(n) - \cos(n+1).$

8. $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n+1}{n}.$

9. Determine the sum of the first n terms of the series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

and the sum of the series.

10. Determine the sum of n terms of the series

$$1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots$$

and the sum of the series.

11. If x is a positive number, determine the sum of n terms of the series

$$1 + e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \cdots$$

and the sum of the series.

By the integral test determine which of the following series converge and which diverge:

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+3n+2}$.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+a^2}}{n^3}$.

19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$.

20. Determine the values of p for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

is convergent and the values for which it is divergent.

21. Determine the values of p for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n(\ln \ln n)^p}$$

is convergent and the values for which it is divergent.

Determine which of the following series converge and which diverge:

22. $\displaystyle\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{n(n+2)} + \cdots$.

23. $\displaystyle\frac{1}{1\cdot 2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \cdots + \frac{1}{n\cdot 2^n} + \cdots$.

24. $\displaystyle\frac{2}{1} + \frac{3}{4} + \frac{4}{9} + \cdots + \frac{n+1}{n^2} + \cdots$.

25. $\displaystyle\frac{2}{7} + \frac{3}{26} + \frac{4}{63} + \cdots + \frac{n}{n^3-1} + \cdots$.

26. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots + \frac{1}{(2n)^2} + \cdots.$

27. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \cdots + \frac{n}{n^2 + 1} + \cdots.$

28. $\frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots.$

29. $\frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{n + 10} + \cdots.$

30. $\frac{1}{2} + \frac{1}{1 + 2^{\frac{3}{2}}} + \cdots + \frac{1}{1 + n^{\frac{3}{2}}} + \cdots.$

31. $\frac{1}{2\cdot 3^2} + \frac{1\cdot 3}{2\cdot 4\cdot 5^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7^2} + \cdots.$

32. $\frac{1}{1} + \frac{1}{4} + \frac{1}{27} + \cdots + \frac{1}{n^n} + \cdots.$

33. $\frac{\ln 2}{4} + \frac{\ln 3}{9} + \cdots + \frac{\ln n}{n^2} + \cdots.$

34. $\frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \cdots + \frac{n}{(n + 1)^2} + \cdots.$

35. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots.$

36. $\frac{1}{3} + \frac{8}{9} + \frac{27}{27} + \cdots + \frac{n^3}{3^n} + \cdots.$

37. $\frac{1}{2} + \frac{1\cdot 3}{2\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8} + \cdots + \frac{1\cdot 3\cdots(2n - 1)}{2\cdot 5\cdots(3n - 1)} + \cdots.$

38. $\frac{1}{2\cdot 2} + \frac{2}{3\cdot 2^2} + \frac{3}{4\cdot 2^3} + \cdots + \frac{n}{(n + 1)2^n} + \cdots.$

39. $\frac{1}{1} + \frac{1\cdot 2}{1\cdot 3} + \cdots + \frac{1\cdot 2\cdots n}{1\cdot 3\cdots(2n - 1)} + \cdots.$

40. $\frac{3}{2^2} + \frac{3^2}{2^3} + \cdots + \frac{3^n}{2^{n+1}} + \cdots.$

41. $\frac{3}{1} + \frac{3^2}{2!} + \cdots + \frac{3^n}{n!} + \cdots.$

42. $\frac{1}{10} + \frac{2!}{10^2} + \frac{3!}{10^3} + \cdots + \frac{n!}{10^n} + \cdots.$

43. By use of the inequalities

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n - 1}$$

determine approximately how many terms of the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

must be added to make the sum greater than 20.

44. Show that Σu_n is convergent if

$$\lim_{n \to \infty} \sqrt[n]{u_n} < 1.$$

45. If a positive series Σu_n is convergent, show that Σu_n^2 is also convergent.

46. If

$$a_n = \frac{(-1)^n}{\sqrt{n}},$$

show that Σa_n is convergent and Σa_n^2 divergent.

Determine the interval of convergence for each of the following series:

47. $1 + \frac{x}{1} + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots.$

48. $\frac{x}{1 \cdot 2} - \frac{x^2}{2 \cdot 2^2} + \frac{x^3}{3 \cdot 2^3} - \frac{x^4}{4 \cdot 2^4} + \cdots.$

49. $1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots.$

50. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$

51. $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots.$

52. $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots.$

53. $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots.$

54. $1 + \frac{x^2}{1} + \frac{1 \cdot 3}{1 \cdot 2} x^4 + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} x^6 + \cdots.$

55. $x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots.$

56. $x - x^4 + x^9 - x^{16} + x^{25} - \cdots.$

57. $1 + \frac{x-1}{2} + \frac{(x-1)^2}{2^2} + \cdots + \frac{(x-1)^n}{2^n} + \cdots.$

58. $1 + \frac{x+1}{1} + \frac{(x+1)^2}{2} + \cdots + \frac{(x+1)^n}{n} + \cdots.$

59. $1 - \frac{x+2}{3} + \frac{(x+2)^2}{3^2} - \frac{(x+2)^3}{3^3} + \cdots.$

60. $1 + \frac{x-3}{1} + \frac{(x-3)^2}{2!} + \cdots + \frac{(x-3)^n}{n!} + \cdots.$

61. $1 - 3(x-2) + 3^2(x-2)^2 - 3^3(x-2)^3 + \cdots.$

62. $1 - \frac{1}{x} + \frac{2}{x^2} - \frac{3}{x^3} + \cdots.$

63. $1 + \frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} + \cdots.$

Assuming that each of the following functions is expressible as a series in powers of x, obtain three non-vanishing terms:

64. $\dfrac{x}{1-x}$.

65. $\sec x$.

66. $\tan x$.

67. $\sqrt{1+x}$.

68. $\frac{1}{2}(e^x + e^{-x})$.

69. $\ln(1-x)$.

70. $\ln\cos x$.

71. $\sin^2 x$.

72. $\sin^{-1} x$.

73. $e^x \sin x$.

In each of the following problems expand $f(x)$ in powers of $x - a$, obtaining four non-vanishing terms:

74. $f(x) = e^x, \quad a = 2$.

75. $f(x) = \ln x, \quad a = 1$.

76. $f(x) = \sqrt{1+x}, \quad a = 3$.

77. $f(x) = \sqrt{x}, \quad a = 4$.

78. $f(x) = \dfrac{1}{x}, \quad a = -1$.

79. $f(x) = x^3, \quad a = 2$.

80. $f(x) = x^{\frac{3}{2}}, \quad a = 1$.

81. $f(x) = \sin x, \quad a = \dfrac{\pi}{6}$.

82. $f(x) = \cos x, \quad a = \dfrac{\pi}{2}$.

83. $f(x) = \sqrt{1-x^3}, \quad x = -2$.

By using an appropriate power series compute each of the following values to three significant figures:

84. $\sin 5°$.

85. $\tan 5°$.

86. $\ln(1.2)$.

87. $\sec 58°$.

88. $\tan 43°$.

89. $\sqrt{e}$.

90. $\tan^{-1}(1.1)$.

91. $\sin^{-1}(0.1)$.

92. Show that $x^{\frac{5}{2}}$ cannot be expressed as a power series in x.

By expanding in Taylor's or Maclaurin's series and showing that the remainder R_n tends to zero as n increases, prove the following expansions for appropriately chosen intervals of convergence:

93. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$.

94. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$.

95. $2^x = 1 + x\ln 2 + \dfrac{(x\ln 2)^2}{2!} + \cdots$.

96. $\ln x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \cdots$.

97. $\dfrac{1}{1+x} = -1 - (x+2) - (x+2)^2 - \cdots$.

98. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$.

99. Estimate the error in the approximation

$$(1.01)^{\frac{5}{2}} = 1 + \tfrac{5}{2}(0.01).$$

100. Show that x, $\sin x$, and $\tan x$ have the same value accurate to 1% if $|x| < 0.1$ radian.

101. For what values of x does $\cos x$ differ from unity by less than 0.0005?

102. For what values of x is the approximation

$$\sin x = x - \frac{x^3}{6}$$

permissible if the allowable error is 0.0001?

103. If $\cos x$ is replaced by $1 - \frac{1}{2}x^2$, for what values of x will the error be less than 1%?

104. By expanding $\ln(1 + x)$ and $\ln(1 - x)$ in powers of x and subtracting, show that

$$\ln \frac{1+x}{1-x} = 2\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

By substituting $x = \frac{1}{3}$ calculate $\ln 2$ to three decimals.

105. Obtain the Maclaurin series for $\sin^2 x$ by using the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

and expanding the right side.

106. Given

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

find an expression for

$$\frac{1}{(1-x)^2}$$

by differentiating both sides.

107. By differentiating the Maclaurin series term by term verify the following:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}e^x = e^x.$$

108. By differentiating

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots,$$

find an expansion for $\sec^2 x$.

109. Given

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

find the Maclaurin expansion for $\ln(1 + x)$ by integrating both sides and determining the constant so that the value is correct when $x = 0$.

110. Expanding

$$\frac{1}{\sqrt{1-t^2}} = (1 - t^2)^{-\frac{1}{2}}$$

by the binomial theorem and integrating, show that

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots.$$

111. Obtain

$$\int_0^x e^{-x^2}\,dx$$

as a series in powers of x.

112. Find

$$\int_0^{\frac{1}{2}} \frac{\sin x}{x}\,dx$$

correct to three decimals by expanding $\sin x$ in a Maclaurin series, dividing by x, and integrating term by term.

113. By use of series find a solution of the equation

$$\tan x = \tfrac{1}{2}.$$

114. By series solve the equation

$$\ln(1 + x) = 1.1.$$

115. By taking the product of the Maclaurin expansions for e^{-x} and $\cos x$ obtain four terms in the expansion of $e^{-x}\cos x$.

116. By squaring

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

and adding, verify the equation

$$\sin^2 x + \cos^2 x = 1.$$

117. Find an expression for

$$\sec x = \frac{1}{\cos x}$$

by expanding $\cos x$ in a Maclaurin series and dividing.

118. By replacing x in

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

by the Maclaurin series for $\tan x$ find

$$e^{\tan x}$$

correct to third powers of x.

119. By writing

$$\frac{1}{1 + x^2} = \frac{1}{2}\,\frac{1}{1 + (x - 1) + \frac{1}{2}(x - 1)^2},$$

performing the division on the right, and integrating term by term, obtain

$$\tan^{-1} x = \int_0^x \frac{dx}{1 + x^2}$$

as a series in powers of $x - 1$ correct to fourth powers of $x - 1$.

CHAPTER XIV

COMPLEX NUMBERS

152. Complex Numbers. In many branches of science and engineering it is found convenient to introduce a symbol

$$i = \sqrt{-1}$$

and to form combinations

$$a + ib$$

of this symbol with real numbers a and b. Such a combination is called a *complex number*, a being its *real part* and ib its *pure imaginary part.*

No value is assigned to i, but algebraic operations involving this symbol are defined as those obtained by following the usual rules of algebra as if i were a real number and replacing i^2 by -1 whenever it occurs.

To each complex number

$$c = a + ib$$

corresponds a *conjugate* complex number

$$\bar{c} = a - ib$$

obtained by replacing i by $-i$. The product

$$c\bar{c} = (a + ib)(a - ib) = a^2 + b^2$$

is real. The conjugate is therefore the multiplier we use to rationalize a complex number. Thus to express the fraction

$$\frac{5 - i}{3 + 2i}$$

as a single complex number we multiply above and below by $3 - 2i$, obtaining

$$\frac{5 - i}{3 + 2i} = \frac{(5 - i)(3 - 2i)}{(3 + 2i)(3 - 2i)} = \frac{13 - 13i}{13} = 1 - i.$$

By expanding and rationalizing denominators any rational combination of complex numbers can be expressed as a single complex number.

In an equation with complex terms real and imaginary parts on the two sides are separately equal. Thus, if

$$a_1 + ib_1 = a_2 + ib_2,$$

we must have

$$a_1 = a_2, \quad b_1 = b_2.$$

For, if this were not so, we could solve for

$$i = \frac{a_1 - a_2}{b_2 - b_1}$$

and thus find a real value for i, which is impossible.

In solving a problem where only a real result has meaning, if all answers are complex, there is no solution. For this reason complex numbers are sometimes called imaginary. In subjects, like alternating currents, where significance has been assigned to complex values, these are, however, entirely real.

Example 1. Show that if

$$r = a + ib$$

is a root of a polynomial equation

$$f(r) = a_0 + a_1r + a_2r^2 + \cdots + a_nr^n = 0$$

with real coefficients, the conjugate

$$\bar{r} = a - ib$$

is also a root.

To show this let

$$f(r) = p + iq$$

be the complex number obtained by replacing r by $a + ib$, expanding, and substituting $i^2 = -1$. If r is replaced by $a - ib$, the same operations give expressions which differ from the former only in having $-i$ in place of i. Thus

$$f(\bar{r}) = p - iq.$$

If r is a root,

$$f(r) = p + iq = 0,$$

p and q are both zero, and therefore

$$f(\bar{r}) = p - iq = 0,$$

which shows that $\bar{r}$ is also a root.

Example 2. Determine whether the parabola $y^2 = 4x$ and the straight line $y = x + 2$ intersect.

Solving simultaneously, we find

$$y = 2 \pm 2i, \quad x = \pm 2i.$$

Since there is no solution with real coördinates, the loci do not intersect.

153. Geometrical Representation. It is customary to represent the complex number $a + ib$ by the vector from the origin to the point of rectangular coördinates a, b. Real numbers ($b = 0$) are represented by points on the x-axis, called the *axis of reals*,

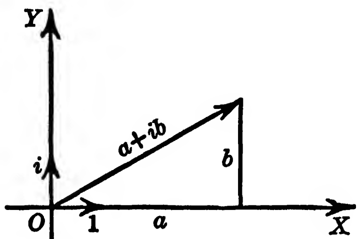

FIGURE 191.

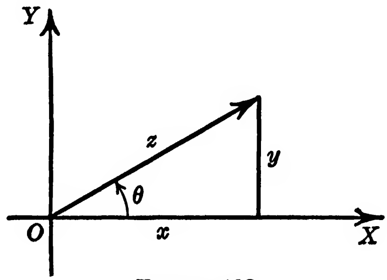

FIGURE 192.

and pure imaginaries ($a = 0$) by points on the y-axis, called the *axis of imaginaries*. The numbers 1 and i are represented by unit vectors along OX and OY respectively.

A complex number

$$z = x + iy \tag{1}$$

has an *absolute value*, or *magnitude*,

$$|z| = \sqrt{x^2 + y^2} = r \tag{2}$$

equal to the length of its representing vector, and an angle θ equal to the angle from the positive direction of the x-axis to this vector. Since this angle may wind any number of times around the origin, it has an infinite number of values differing by multiples of 2π.

In terms of magnitude and angle we have

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{3}$$

and therefore

$$z = r(\cos\theta + i\sin\theta). \tag{4}$$

We sometimes refer to this as the polar form of the complex number z.

The coefficients x_1, y_1 and x_2, y_2 in the complex numbers

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

are the components of the vectors that represent these numbers. Since corresponding coefficients are added and subtracted in the combinations

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2),$$

$$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2),$$

it follows that the sum and difference of two complex numbers

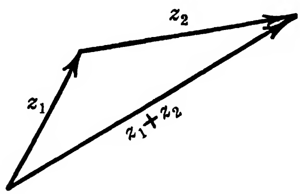

FIGURE 193.

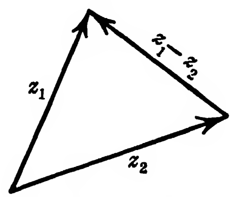

FIGURE 194.

are graphically obtained by adding and subtracting the vectors that represent them.

Similarly, the vector representing $z_1 + z_2 + \cdots + z_n$ is the sum of the vectors representing its parts. Since a straight line is the

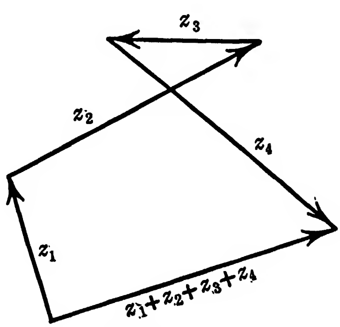

FIGURE 195.

shortest distance between two points, these constructions show at once that

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|, \tag{5}$$

that is, *the absolute value of a sum of complex numbers is equal to or less than the sum of the absolute values of its parts.*

154. Product and Quotient. By direct multiplication of two complex numbers

$$z_1 = r_1 (\cos\theta_1 + i \sin\theta_1),$$
$$z_2 = r_2 (\cos\theta_2 + i \sin\theta_2) \tag{1}$$

we find that their product

$$z_1 z_2 = r_1 r_2[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)]$$
$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \tag{2}$$

has the magnitude $r_1 r_2$ and the angle $\theta_1 + \theta_2$. Thus *the product of two complex numbers is obtained by multiplying their magnitudes and adding their angles.*

To find the quotient of two complex numbers

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i \sin\theta_1)}{r_2(\cos\theta_2 + i \sin\theta_2)}$$

we rationalize the denominator and thus get

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \tag{3}$$

The quotient of two complex numbers is obtained by dividing their magnitudes and subtracting their angles.

The above equations show that multiplication and division by a complex number can be graphically interpreted as operations of rotation and stretching. Thus (2) shows that multiplication by z_2 has the effect of rotating z_1 through an angle equal to the angle of z_2 and stretching it in a ratio equal to the magnitude of z_2. Similarly division by z_2 is equivalent to rotating backward through an angle equal to the angle of z_2 and shrinking in a ratio equal to the magnitude of z_2. Multiplication by -1, for example, is equivalent to rotation through 180°, and division by i produces rotation backward through 90°.

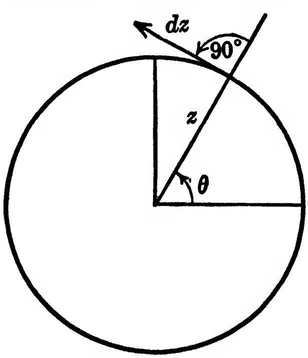

FIGURE 196.

Example. If z represents the vector from the origin to a point moving around a circle with center at the origin, find

$$\frac{dz}{z}.$$

Graphically dz is tangent to the circle and directed 90° ahead of z. Also the magnitude of the above fraction is

$$\left|\frac{dz}{z}\right| = \frac{|dz|}{|z|} = \frac{ds}{r} = d\theta.$$

Thus

$$\frac{dz}{z} = i\,d\theta.$$

The same result is obtained if we write

$$z = r(\cos\theta + i\sin\theta),$$

$$dz = r(-\sin\theta + i\cos\theta)\,d\theta,$$

and determine the quotient

$$\frac{dz}{z} = \frac{-\sin\theta + i\cos\theta}{\cos\theta + i\sin\theta}\,d\theta = i\,d\theta.$$

155. Powers and Roots. If n is a positive integer, the nth power of a complex number

$$z = r(\cos\theta + i\sin\theta) \tag{1}$$

is the product of n factors z. Its magnitude is thus the product of n factors r, and its angle the sum of n angles θ. Therefore

$$z^n = r^n(\cos n\theta + i\sin n\theta). \tag{2}$$

Elimination of z from (1) and (2) gives

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{3}$$

which is called *DeMoivre's theorem.* By expanding the left side and separately equating reals and imaginaries, it determines expressions for $\cos n\theta$ and $\sin n\theta$. For example, if $n = 2$, the equation becomes

$$\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta,$$

whence

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

An nth root of z must satisfy the equation

$$\left(z^{\frac{1}{n}}\right)^n = z = r(\cos\theta + i\sin\theta).$$

Its magnitude is therefore $r^{\frac{1}{n}}$ and its angle is such that after multiplication by n the result is θ or differs from θ by a multiple of 2π.

Thus

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right], \tag{4}$$

where k is any integer. The values determined by $k = 0, 1, 2, \cdots, n - 1$ are distinct, but other values of k give roots already found.

A complex number different from zero therefore has n distinct nth roots. These all have the same magnitude but their angles differ by multiples of $\frac{2\pi}{n}$. If vectors representing these roots are drawn from a common origin, their ends are vertices of a regular polygon of n sides.

Example 1. Solve the equation $z^3 = 8$.

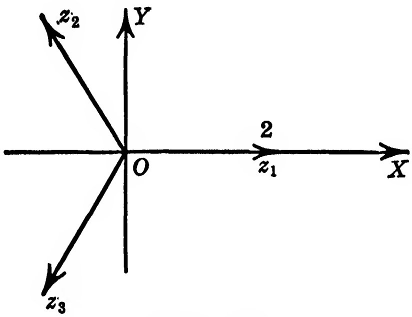

FIGURE 197.

One root is $z = 2$. The other two have the same magnitude and form with this angles of 120° and 240°. The three roots are therefore

$$\begin{aligned} z_1 &= 2, \\ z_2 &= 2(\cos 120° + i\sin 120°) = -1 + i\sqrt{3}, \\ z_3 &= 2(\cos 240° + i\sin 240°) = -1 - i\sqrt{3}. \end{aligned}$$

Example 2. Find the value of

$$\sqrt{2 + 3i}.$$

The number $2 + 3i$ is represented by a vector of length

$$r = \sqrt{13}$$

and angle

$$\theta = \tan^{-1}\tfrac{3}{2} = 56°\,19'.$$

Its square roots therefore have the magnitude

$$\sqrt{r} = (13)^{\frac{1}{4}}$$

and the angles

$$\theta_1 = 28° \, 9.5', \quad \theta_2 = 208° \, 9.5'.$$

Thus

$$\sqrt{2 + 3i} = \pm r^{\frac{1}{2}}[\cos \theta_1 + i \sin \theta_1] = \pm[1.674 + 0.896i].$$

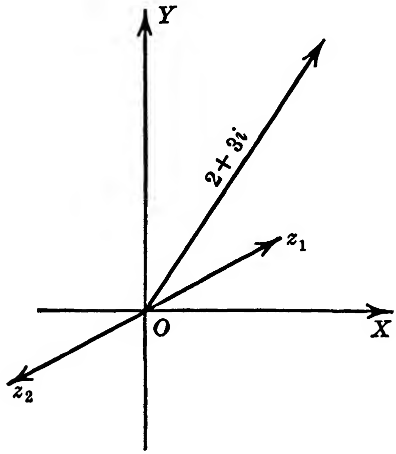

FIGURE 198.

156. Limit, Function, Continuity. A complex number z which varies through a discrete or continuous range of values is said to approach, or tend, to a complex number a as limit if

$$\lim |z - a| = 0,$$

and to have an infinite limit, or tend to infinity, or become infinite, if

$$|z| \rightarrow \infty.$$

We write

$$w = f(z) \tag{1}$$

and say that w is a function of a complex variable z on a certain range, if to each value of z on that range corresponds one or more values of w. The function is called continuous at $z = a$ if it tends to a limit as z tends to a.

With these definitions practically all that has been said about limits and continuity for real variables is applicable to complex variables. It is only relations involving the notions greater than and less than that are not extensible.

The derivative of $f(z)$ is defined by the equation

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \tag{2}$$

provided that the limit on the right exists. In case of algebraic functions the expressions used, and so the final differentiation formulas, have exactly the same form whether the variables are real or complex. For example

$$\frac{d}{dz} z^n = nz^{n-1}. \tag{3}$$

A similar remark is applicable to integrals expressible by algebraic functions.

157. Power Series. A series of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots, \tag{1}$$

where $a_0, a_1, a_2, \cdots, a_n \cdots$ are constants and

$$z = x + iy,$$

is called a power series. Similarly

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + \cdots \tag{2}$$

is also a power series which has values distributed about $z = a$ in the same way the values of (1) are distributed about $z = 0$.

If we consider the symbols complex, the proofs in §148 show that if (2) converges at $z - a = b$ it converges absolutely for all values $|z - a| < b$, and if it diverges at $z - a = b$ it diverges for all values $|z - a| > b$. The region of convergence is therefore of the form

$$|z - a| < R.$$

It is thus in general a circle with center $z = a$ but in special cases may shrink to a point or extend to infinity. This is called the *circle of convergence* of the series. On the circle of convergence the series may converge at all points, diverge at all points, or converge at some and diverge at others. The radius of the convergence circle is usually found by the ratio test.

Example. Find the circle of convergence of the series

$$(z-1) + \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \cdots + \frac{(z-1)^n}{n} + \cdots.$$

The ratio of consecutive terms is

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{n}(z-1),$$

and the limit of its absolute value is

$$L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = |z - 1|.$$

The series therefore converges if

$$|z - 1| < 1$$

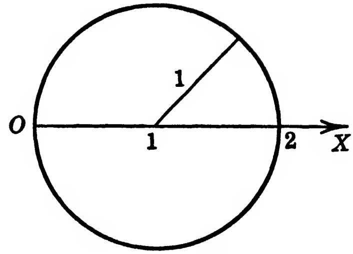

FIGURE 199.

and diverges if $|z - 1| > 1$. The circle of convergence (Figure 199) thus has the center $z = 1$ and radius $r = 1$. The series converges at all points inside this circle and diverges at all points outside. On the circle it converges conditionally at $z = 0$ and diverges at $z = 2$.

158. Exponential Function. For any real value of z (§150, example 1)

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots. \tag{1}$$

This series converges not only for all real values but consequently also for all complex values of z. When z is complex, e^z cannot be defined as a power of e, for raising a number to an imaginary power does not make sense. When z is complex we therefore define e^z as the value determined by this series.

By multiplying the series for e^{z_1} by that for e^{z_2} it can be easily verified that the function thus defined has the exponential property

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2} \tag{2}$$

which the notation suggests.

If θ is a real number and $z = i\theta$, we have

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right).$$

The series in parentheses represent $\cos\theta$ and $\sin\theta$ (Problems 93 and 94, page 323). We thus have the remarkable relation

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3}$$

called *Euler's formula.*

A similar equation

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{4}$$

is obtained by replacing θ by $-\theta$. By taking the sum and difference of the two equations, we get

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{5}$$

Thus the sine and cosine, and consequently all the trigonometric functions, are expressible in terms of exponentials. From these expressions and the properties of exponents all the identities connecting trigonometric functions can be obtained.

In virtue of Euler's formula the polar expression for a complex number

$$z = r(\cos\theta + i\sin\theta)$$

can be written

$$z = re^{i\theta}. \tag{6}$$

When complex numbers are represented in this form the expressions for product, quotient, and power become obvious consequences of the exponent formula (2).

Example. Determine the components of acceleration of a moving particle in polar coördinates.

Let

$$z = re^{i\theta}$$

represent the vector from the origin to the particle at time t. Its velocity is then

$$\frac{dz}{dt} = e^{i\theta}\frac{dr}{dt} + ire^{i\theta}\frac{d\theta}{dt}$$

and its acceleration is

$$\frac{d^2z}{dt^2} = e^{i\theta}\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] + ie^{i\theta}\left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right].$$

Since $e^{i\theta}$ represents a unit vector along the radius and $ie^{i\theta}$ a unit vector perpendicular to the radius and extending in the direction of increasing θ, the expressions in brackets are the components required.

159. Plane Motion. A rigid body is said to have *plane motion* when all points of the body move parallel to a fixed plane. Such a motion is evidently determined by that of a section parallel to

the fixed plane. In Figure 200 is shown such a section moving in its plane.

In general the motion involves rotation which turns all lines of the figure through the same angle. If θ is the angle through which the figure has rotated at time t,

$$\omega = \frac{d\theta}{dt} \tag{1}$$

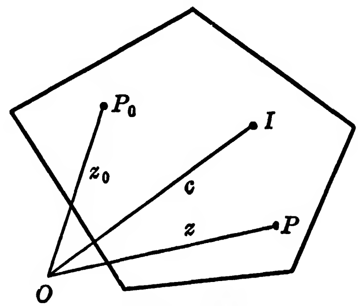

FIGURE 200.

is the angular velocity at that time. Both θ and ω are considered positive in the counterclockwise direction. Let P_0 be a definite point of the figure, P any other point, and let z_0 and z be the complex numbers representing vectors from the origin to those points. As the figure moves

$$z - z_0 = \overrightarrow{P_0P} \tag{2}$$

rotates with angular velocity ω but does not change length. Thus

$$\frac{d}{dt}(z - z_0) = i\omega(z - z_0). \tag{3}$$

The velocity of P is

$$v = \frac{dz}{dt}. \tag{4}$$

Substituting for $\frac{dz}{dt}$ its value from (3), we have

$$v = \frac{dz_0}{dt} + i\omega(z - z_0). \tag{5}$$

If ω is not zero, this can be written

$$v = i\omega(z - c), \tag{6}$$

where

$$c = z_0 + \frac{i}{\omega}\frac{dz_0}{dt}. \tag{7}$$

Equation (6) gives the velocity of any point in the body at time t. At an instant when ω is not zero it shows that the point

I, where $z = c$, has zero velocity and that every other point has the velocity

$$v = i\omega \overrightarrow{IP}$$

it would have in a steady rotation with angular velocity ω about I as center. This point I is called the *instantaneous center of rotation.* It usually changes with the time. At an instant when $\omega = 0$ all points in the body have the same velocity and there is no instantaneous center of rotation.

The instantaneous center is usually found either by directly locating the point I with zero velocity or by using the fact that $\overrightarrow{IP}$ is perpendicular to the direction of motion of P.

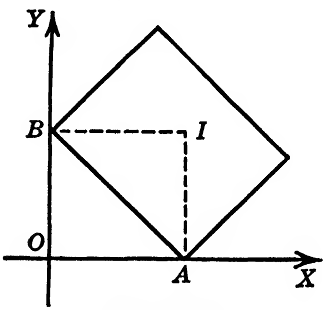

FIGURE 201.

Example. A body moves with one point A in the x-axis, another B in the y-axis. Find the instantaneous center of rotation.

Since A moves along the x-axis, the instantaneous center is on the line perpendicular to the x-axis at A. Similarly, it is on the line perpendicular to the y-axis at B. It is therefore the point I (Figure 201) where these perpendiculars intersect.

160. Logarithm. The logarithm of z is defined as a number

$$w = \ln z \tag{1}$$

such that

$$e^w = z. \tag{2}$$

When z is real and positive, a real value can be found for w. When

$$z = r(\cos\theta + i\sin\theta)$$

is negative or complex, the logarithm is a complex number

$$w = u + iv, \tag{3}$$

which satisfies the equation

$$e^w = e^{u+iv} = e^u(\cos v + i\sin v) = r(\cos\theta + i\sin\theta).$$

Thus

$$e^u\cos v = r\cos\theta, \quad e^u\sin v = r\sin\theta, \tag{4}$$

and, since u is real,

$$e^u = r, \quad u = \ln r, \quad v = \theta + 2n\pi, \tag{5}$$

where n is any integer.

Therefore

$$\ln z = u + iv = \ln r + i(\theta + 2n\pi) \tag{6}$$

has infinitely many values differing by multiples of $2\pi i$. In particular, when $z = -1$, $\theta = \pi$ and hence

$$\ln(-1) = (2n+1)\pi i. \tag{7}$$

By differentiating the series for e^z term by term we find

$$\frac{d}{dz} e^z = e^z. \tag{8}$$

Similarly, by differentiating both sides of (2) we get

$$e^w \frac{dw}{dz} = 1, \quad \frac{dw}{dz} = \frac{1}{e^w},$$

and therefore

$$\frac{d}{dz} \ln z = \frac{1}{z}. \tag{9}$$

Thus the derivatives of e^z and $\ln z$ are determined by the same formulas, (8) and (9), whether z is real or complex.

Example. Find the value of

$$\int_0^1 \frac{dx}{x+i}.$$

Integrating as if the values were real, we have

$$\int_0^1 \frac{dx}{x+i} = [\ln(x+i)]_0^1 = \ln(1+i) - \ln i.$$

To interpret this we must consider

$$\ln(x+i) = \ln\sqrt{x^2+1} + i\theta$$

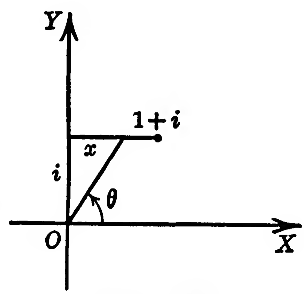

FIGURE 202.

as varying continuously while x varies from 0 to 1. If we take $\theta = \frac{\pi}{2}$ when $x = 0$ (Figure 202), we must then take $\theta = \frac{\pi}{4}$ when $x = 1$. Thus

$$\int_0^1 \frac{dx}{x+i} = \ln\sqrt{2} + i\frac{\pi}{4} - i\frac{\pi}{2} = \frac{1}{2}\ln 2 - i\frac{\pi}{4}.$$

161. Hyperbolic Functions. By omitting the symbol i in the exponential expressions for sine and cosine [§158, (5)] we obtain two functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{1}$$

called the *hyperbolic sine* and *hyperbolic cosine* of x.

In terms of these, by equations analogous to those in trigonometry, we define the hyperbolic tangent, cotangent, secant, and cosecant, as the functions

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{2}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \tag{3}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{4}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}. \tag{5}$$

Graphs of the most important of these functions are shown in Figure 203. It is to be noted that $\sinh x$ takes all real values,

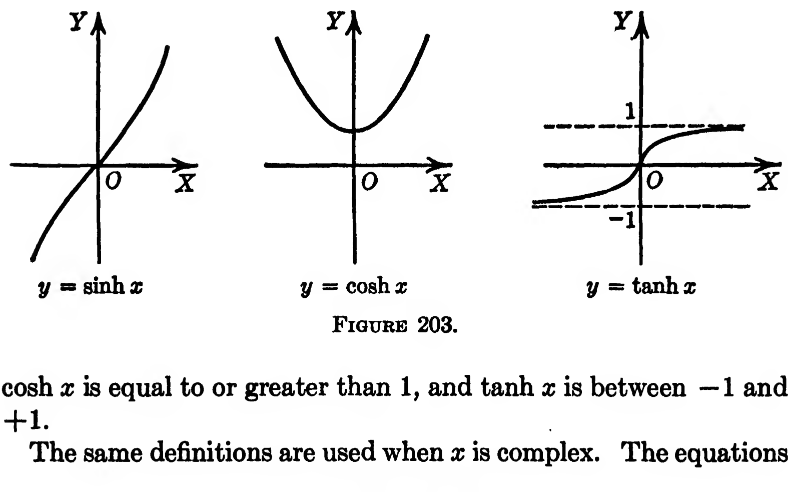

FIGURE 203.

$\cosh x$ is equal to or greater than 1, and $\tanh x$ is between -1 and $+1$.

The same definitions are used when x is complex. The equations

$$\sinh (ix) = \frac{e^{ix} - e^{-ix}}{2} = i \sin x, \tag{6}$$

$$\cosh (ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x, \tag{7}$$

express the hyperbolic functions of a pure imaginary as trigonometric functions of a real number, and the equations

$$\sin (ix) = \frac{e^{-x} - e^{x}}{2i} = i \sinh x, \tag{8}$$

$$\cos (ix) = \frac{e^{-x} + e^{x}}{2} = \cosh x, \tag{9}$$

make the corresponding transformations from trigonometric to hyperbolic functions.

The hyperbolic functions satisfy identities similar to those connecting trigonometric functions. Thus, by squaring and subtracting we find

$$\cosh^2 x - \sinh^2 x = 1. \tag{10}$$

The only difference between these and the corresponding trigonometric identities consists in certain differences in algebraic sign, as in the equation just written. These differences in sign are due to the factor i on the right side of equations (6) and (8). When two sines are multiplied together (or two functions, such as tangents, which may be considered as having a sine as factor) the factor i^2 is replaced by -1 and the corresponding hyperbolic formula has a different sign. Thus

$$\cos (x + y) = \cos x \cos y - \sin x \sin y,$$

$$\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y.$$

By direct use of the definitions we find

$$\frac{d}{dx} \sinh u = \cosh u \frac{du}{dx}, \tag{11}$$

$$\frac{d}{dx} \cosh u = \sinh u \frac{du}{dx}, \tag{12}$$

$$\frac{d}{dx} \tanh u = \operatorname{sech}^2 u \frac{du}{dx}, \tag{13}$$

$$\frac{d}{dx} \coth u = -\operatorname{csch}^2 u \frac{du}{dx}, \tag{14}$$

$$\frac{d}{dx} \operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}, \tag{15}$$

$$\frac{d}{dx} \operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}. \tag{16}$$

In engineering applications hyperbolic functions are analogous to the trigonometric, the occurrence of the one type of function or the other depending on the algebraic signs and magnitudes of the physical constants.

For example, if a particle moving along the x-axis is *attracted toward* the origin with a force proportional to the distance, its position at time t is

$$x = a \sin (bt + c),$$

where a, b, c are constant. If it is *repelled* with a force proportional to the distance, its position is

$$x = a \sinh (bt + c).$$

162. Geometrical Representation of Hyperbolic Functions. The name hyperbolic is given to these functions because they are related to the rectangular hyperbola

$$x^2 - y^2 = 1$$

in the same way the trigonometric functions are related to the unit circle

$$x^2 + y^2 = 1.$$

In case of the trigonometric functions,

$$x = \cos \theta, \quad y = \sin \theta \tag{1}$$

are coördinates of a point P on the circle. The angle θ associated

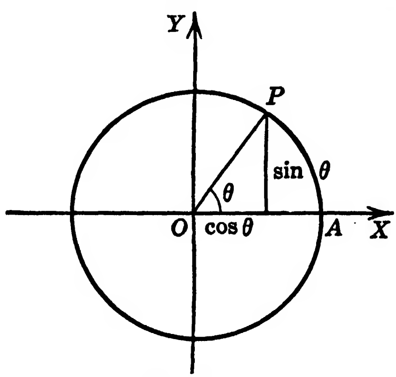

FIGURE 204.

with this point can be defined in two ways. In terms of the arc $s = AP$, it has for differential

$$d\theta = \frac{ds}{r}, \tag{2}$$

where r is the radius OP. Also the area of the sector AOP is $\frac{1}{2}\theta$, and hence

$$\theta = 2 \text{ (area of sector } AOP). \tag{3}$$

Similarly,

$$x = \cosh u, \quad y = \sinh u \tag{4}$$

satisfy the equation

$$x^2 - y^2 = \cosh^2 u - \sinh^2 u = 1,$$

and are therefore coördinates of a point P on the hyperbola.

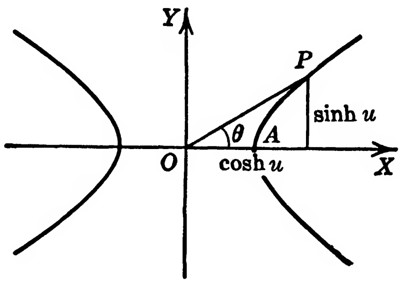

FIGURE 205.

The arc $s = AP$ of the hyperbola has for differential

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\sinh^2 u + \cosh^2 u}\, du = r\, du,$$

where r is the radius OP. Hence

$$du = \frac{ds}{r} \tag{5}$$

as in the circle. Also the area of the sector AOP is

$$\tfrac{1}{2}xy - \int_1^x y\, dx = \int \tfrac{1}{2}\, d(xy) - y\, dx = \int \tfrac{1}{2}(x\, dy - y\, dx)$$

$$= \int \tfrac{1}{2}(\cosh^2 u - \sinh^2 u)\, du = \tfrac{1}{2}u.$$

Therefore

$$u = 2 \text{ (area of sector } AOP). \tag{6}$$

Thus, whether u is determined by using the differential of arc or area of sector, $\sinh u$ and $\cosh u$ are related to the hyperbola in the same way that $\sin \theta$ and $\cos \theta$ are related to the circle.

163. Inverse Hyperbolic Functions. To the hyperbolic functions correspond inverse functions defined by the equations

$$y = \sinh^{-1} x, \quad \text{if} \quad x = \sinh y,$$

$$y = \cosh^{-1} x, \quad \text{if} \quad x = \cosh y,$$

$$y = \tanh^{-1} x, \quad \text{if} \quad x = \tanh y,$$

and similar equations for the other functions.

These inverse functions are also expressible as logarithms. From the equation

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

for example, we have

$$e^{2y} - 2xe^y - 1 = 0,$$

whence

$$e^y = x + \sqrt{x^2 + 1},$$

the positive sign of the radical being used because e^y is positive. Thus

$$y = \sinh^{-1} x = \ln (x + \sqrt{x^2 + 1}). \tag{1}$$

In a similar way we find

$$\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}), \quad x \geqq 1, \tag{2}$$

$$\tanh^{-1} x = \tfrac{1}{2} \ln \frac{1 + x}{1 - x}, \quad |x| < 1, \tag{3}$$

$$\coth^{-1} x = \tfrac{1}{2} \ln \frac{x + 1}{x - 1}, \quad |x| > 1. \tag{4}$$

From these equations we obtain the following differentiation and integration formulas:

$$\frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}. \tag{5}$$

$$\frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \quad u > 1. \tag{6}$$

$$\frac{d}{dx} \tanh^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx}, \quad |u| < 1. \tag{7}$$

$$\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1. \tag{8}$$

$$\frac{d}{dx}\operatorname{sech}^{-1} u = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1. \tag{9}$$

$$\frac{d}{dx}\operatorname{csch}^{-1} u = -\frac{1}{u\sqrt{1+u^2}}\frac{du}{dx}. \tag{10}$$

$$\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\frac{u}{a} + C. \tag{11}$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\frac{u}{a} + C, \quad u > a. \tag{12}$$

$$\int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C, \quad u^2 < a^2. \tag{13}$$

$$\int \frac{du}{u^2-a^2} = -\frac{1}{a}\coth^{-1}\frac{u}{a} + C, \quad u^2 > a^2. \tag{14}$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\frac{u}{a} + C, \quad 0 < u < a. \tag{15}$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\frac{u}{a} + C. \tag{16}$$

In these formulas it is assumed that the *inverse hyperbolic cosine and inverse hyperbolic secant are positive.* Otherwise the algebraic signs must be changed.

Numerical values of the hyperbolic functions and other integration formulas are found in mathematical tables.

PROBLEMS

Reduce each of the following expressions to a single complex number $a + bi$:

1. $\dfrac{1+i}{1-i}$. **2.** $\dfrac{2+3i}{3-2i}$.

3. $(2-i)^3$. **4.** $(2-3i)(3+4i)$.

5. Given

$$f(z) = z^2 + z + 1,$$

find $f(3+2i)$ and $f(3-2i)$.

6. Given $z = x + iy$,

$$f(z) = \frac{1 + 3z}{1 - z},$$

show that

$$f(\bar{z}) = \overline{f(z)}.$$

7. Show that

$$|z_1 z_2| = |z_1| \cdot |z_2|, \qquad \left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|}.$$

8. Show that $z = 1 + i$ is a solution of the equation

$$z^3 - 3z^2 + 4z = 2$$

and that $z = 1 - i$ is also a solution.

9. Show that $z = 1 + i$ is a solution of the equation

$$z^3 - z^2 + 2z = 2i$$

but that $z = 1 - i$ is not a solution. Explain.

10. If x, y satisfy the equation of a curve, and are not real, (x, y) is called an imaginary point on the curve. If distance is calculated by the usual formula, show that the imaginary points in which a parabola intersects its directrix are at zero distance from the focus.

11. If $A(1, -1)$, $B(3, 1)$, $C(2, 4)$ are the vertices of a triangle, determine the complex numbers which represent $\overrightarrow{AB}$, $\overrightarrow{BC}$ and show that their sum represents $\overrightarrow{AC}$.

12. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, show that $|z_2 - z_1|$ is the distance between (x_1, y_1) and (x_2, y_2).

13. If

$$\frac{a}{b} = p + iq$$

where a and b are complex, p and q real numbers, show that $p|b|$ and $q|b|$ are the components of a along and perpendicular to b.

Express each of the following in the polar form $r(\cos\theta + i\sin\theta)$:

14. -2.

15. $3i$.

16. $2 + 2i$.

17. $4i - 4$.

18. $1 + \sqrt{3}i$.

19. $3\sqrt{3} + 3i$.

Given $z_1 = 2(\cos 60° + i\sin 60°)$, $z_2 = 4(\cos 120° + i\sin 120°)$, express each of the following as a complex number $a + ib$:

20. z_1^2.

21. $z_1 z_2$.

22. $\dfrac{z_2}{z_1}$.

23. $z_2 - z_1$.

Determine the two square roots of each of the following complex numbers

24. $2i$.

25. $2 + 2\sqrt{3}i$.

26. $1 - \sqrt{3}i$.

27. $1 + i$.

28. If z_1, z_2, z_3, z_4 are the complex numbers representing vectors from the origin to four points on a circle, show that

$$\frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}$$

is a real number.

29. By DeMoivre's theorem find $\sin 3\theta$ and $\cos 3\theta$.

30. By DeMoivre's theorem find $\sin 4\theta$ and $\cos 4\theta$.

Solve the following equations:

31. $z^3 + 8 = 0$.

32. $z^2 + z + 1 = 0$.

33. $z^4 - 16 = 0$.

34. $z^4 + 6z^2 + 5 = 0$.

35. $z^4 - z^2 + 1 = 0$.

36. $z^6 + 1 = 0$.

37. If

$$u = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$

show that the n roots of the equation $z^n = 1$ are $1, u, u^2 \cdots, u^{n-1}$ and that

$$1 + u + u^2 + \cdots + u^{n-1} = 0.$$

38. The equation

$$z = w - \frac{a}{3w}$$

makes two values of w correspond to each value of z. Show that the six values of w that correspond to roots of the equation

$$z^3 + az + b = 0$$

are graphically represented by two equilateral triangles.

Find the circles of convergence of the following series:

39. $1 + \dfrac{z}{2} + \dfrac{z^2}{2^2} + \cdots + \dfrac{z^n}{2^n} + \cdots$.

40. $z - 3 + \dfrac{(z-3)^2}{2^2} + \cdots + \dfrac{(z-3)^n}{n^2} + \cdots$.

41. $1 + \dfrac{z-2}{3} + \dfrac{(z-2)^2}{3^2} + \cdots + \dfrac{(z-2)^n}{3^n} + \cdots$.

42. $1 - (z-1) + \dfrac{(z-1)^2}{2^2} - \dfrac{(z-1)^3}{3^3} + \cdots$.

43. $1 + \dfrac{(z+2)}{1\cdot 2} + \dfrac{(z+2)^2}{2\cdot 3} + \cdots + \dfrac{(z+2)^n}{n(n+1)} + \cdots$.

44. $1 + z + 2^2z^2 + 3^3z^3 + \cdots + n^nz^n + \cdots$.

45. $\dfrac{z-2}{1} + \dfrac{1}{2}\dfrac{(z-2)^3}{3} + \dfrac{1}{2}\cdot\dfrac{3}{4}\dfrac{(z-2)^5}{5} + \cdots$.

46. $1 + \dfrac{(z-a)^2}{2!} + \dfrac{(z-a)^4}{4!} + \dfrac{(z-a)^6}{6!} + \cdots$.

47. Show that if a power series converges for all real values of a variable z it converges for all complex values of z.

48. Show that since

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

becomes infinite at $z = i$ the series cannot converge when $|z| > 1$.

49. Show that since the logarithm of 0 is infinite the series

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

cannot converge when $|z| > 1$.

50. Show that if m is not a positive integer a derivative of $(1+x)^m$ becomes infinite at $x = -1$ and therefore that the binomial series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

cannot converge if $|x| > 1$.

51. Show that

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

52. By expressing the trigonometric functions in terms of exponentials verify the equation

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

for complex values of x and y.

Express the following as complex numbers:

53. $e^{\pi i}$.

54. $e^{-\frac{\pi}{3}i}$.

55. $e^{2-\frac{\pi}{2}i}$.

56. e^i.

57. Find the numbers r and θ such that

$$1 + 2i = re^{i\theta}.$$

58. A wheel of radius a rolls with angular velocity ω along the x-axis. Find the instantaneous center of rotation and the velocities of the two points at the ends of a horizontal diameter. Note that the angular velocity is positive when the rotation is counterclockwise.

59. The center of a wheel moves with velocity v along the x-axis while the wheel rotates with angular velocity ω. Find the instantaneous center of rotation.

60. A piston A moving along a line CA is connected by a rod AB to a crank CB rotating about C. Find the instantaneous center of rotation of AB.

61. A rigid body is in plane motion with angular velocity ω and angular acceleration

$$\alpha = \frac{d\omega}{dt}.$$

Using the notation of §159, determine the acceleration of the point P in the body. At an instant when ω and α are not both zero show that all points in the body have the accelerations they would have in a rotation with angular velocity ω and angular acceleration α about a certain axis. This axis is called

the *instantaneous axis of acceleration.* In a plane figure the intersection of the plane and axis of acceleration is called the instantaneous center of acceleration

62. A wheel of radius a rolls along a horizontal line with angular velocity ω and angular acceleration α. Find the instantaneous center of acceleration.

63. A wheel of radius a and center C rolls on the outside of a fixed wheel of radius b and center O, the line OC rotating with constant angular velocity. Find the instantaneous centers of rotation and acceleration.

64. A plane figure is in plane motion. From each point P of the figure a vector is drawn equal to the velocity of P. Show that the ends of these vectors at each instant form a similar figure, if ω is not zero.

65. A rod rolls with constant angular velocity ω around a circle of radius a. A particle, starting at the point of contact, moves along the rod with speed v. By means of a complex number represent the position of the particle at time t. Take the origin at the center of the circle and the axis of reals through the starting point.

66. Show that

$$\ln (z_1 z_2) = \ln z_1 + \ln z_2 + 2k\pi i,$$

where k is an integer or zero.

67. If we take a particular value of

$$\ln z = \ln r + i(\theta + 2k\pi)$$

and let this vary continuously as z moves in the positive direction around a circle with center at the origin, what does this value become when z returns to its initial position?

68. By writing

$$\frac{1}{1 + x^2} = \frac{i}{2}\left[\frac{1}{x + i} - \frac{1}{x - i}\right]$$

and integrating, check the equation

$$\int_0^1 \frac{dx}{1 + x^2} = \frac{\pi}{4}.$$

69. If x_1, x_2, x_3 are the three roots of the equation $x^3 + 1 = 0$, show that

$$\frac{1}{x^3 + 1} = -\frac{1}{3}\left[\frac{x_1}{x - x_1} + \frac{x_2}{x - x_2} + \frac{x_3}{x - x_3}\right].$$

Hence determine

$$\int_0^1 \frac{dx}{x^3 + 1}.$$

70. Prove the following identities:

(*a*) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

(*b*) $\sinh 2x = 2 \sinh x \cosh x$.

(*c*) $2 \cosh^2 x = \cosh (2x) + 1$.

(*d*) $2 \sinh^2 x = \cosh (2x) - 1$.

In the following problems find the other hyperbolic functions:

71. $\sinh x = \frac{4}{3}$.

72. $\cosh x = \frac{5}{4}$.

73. $\tanh x = \frac{5}{13}$.

74. $\operatorname{sech} x = 0.8$.

75. If $\sinh x = \frac{4}{3}$ and $\sinh y = \frac{3}{4}$, find $\sinh (x + y)$.
76. If $\tanh x = \frac{3}{4}$, find $\tanh 2x$.
77. Show that

$$(\cosh \theta + \sinh \theta)^n = \cosh n\theta + \sinh n\theta.$$

Make graphs of the following equations:

78. $y = \coth x$.
79. $y = \operatorname{sech} x$.
80. $y = \operatorname{csch} x$.
81. $y = \sinh^2 x$.

By direct use of the definitions find the derivatives of the following functions:

82. $\operatorname{sech} x$.
83. $\operatorname{csch} x$.
84. $\coth x$.
85. $\operatorname{sech}^{-1} x$.
86. $\coth^{-1} x$.
87. $\operatorname{csch}^{-1} x$.

88. Derive equation (3), §163.
89. Derive equation (4), §163.

Integrate the following:

90. $\displaystyle\int \sinh^3 x\, dx.$
91. $\displaystyle\int \tanh^2 x\, dx.$
92. $\displaystyle\int \frac{dx}{x\sqrt{1 - x^2}}.$
93. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + 1}}.$

By use of tables evaluate the following integrals:

94. $\displaystyle\int_0^3 \frac{dx}{\sqrt{x^2 + 9}}.$
95. $\displaystyle\int_2^3 \frac{dx}{x^2 - 16}.$
96. $\displaystyle\int_5^\infty \frac{dx}{x^2 - 16}.$
97. $\displaystyle\int_0^2 \frac{dx}{9 - x^2}.$

98. If θ belongs to the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and

$$\sinh u = \tan \theta,$$

then

$$\theta = \operatorname{gd} u = \tan^{-1} (\sinh u)$$

is called the gudermannian of u. Show that $\cosh u = \sec \theta$, $\tanh u = \sin \theta$, $\coth u = \csc \theta$, $\operatorname{csch} u = \cot \theta$, $\operatorname{sech} u = \cos \theta$. By means of this function work involving hyperbolic functions can be handled with trigonometric functions and conversely.

99. In Figure 205 show that

$$2\theta = \operatorname{gd} (2u).$$

100. Show that

$$\int \sec x\, dx = \operatorname{gd}^{-1} x.$$

CHAPTER XV

SPACE COÖRDINATES AND VECTORS

164. Rectangular Coördinates. A rotation about a directed axis is called positive, or *right-handed*, if it causes a right-handed screw to advance in the positive direction along the axis (Figure 206).

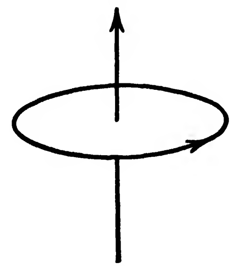

FIGURE 206.

As a reference system for rectangular coördinates in space we use right-handed axes. By this we mean a set of three mutually perpendicular scales $X'X$, $Y'Y$, $Z'Z$ with their zero points coincident at O and their positive directions so related that OX goes into OY by a positive rotation of 90° about OZ.

If planes perpendicular to the axes are passed through a point P, the numbers x, y, z in the scales at the points of intersection are called the rectangular, or Cartesian, coördinates of P. The point with coördinates

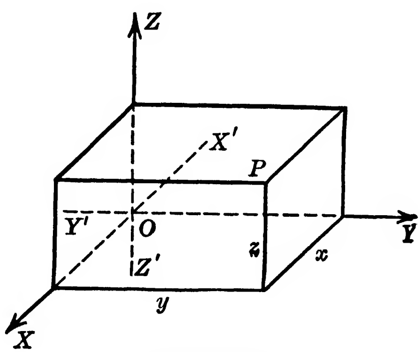

FIGURE 207.

x, y, z is represented by the notation (x, y, z), and the notation $P(x, y, z)$ is used to indicate that P has the coördinates x, y, z.

Taken two at a time the axes determine three planes called the xy-plane, yz-plane, zx-plane in terms of the axes included. On each of these planes one coördinate is zero, and on each axis two coördinates are zero.

165. Vectors. The vectors of unit length and positive direction along OX, OY, OZ are denoted by **i**, **j**, **k** respectively. The symbol **i** (bold-faced type) for a unit vector should be distinguished from the imaginary unit $i = \sqrt{-1}$.

The point P with coördinates x, y, z is often represented by the vector

$$\mathbf{r} = \overrightarrow{OP} \tag{1}$$

from the origin to P. This is the sum of three vectors $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BP}$, Figure 208, equal to $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k}$, respectively. Hence

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{2}$$

The numbers x, y, z are the components of $\mathbf{r}$ along the coördinate axes.

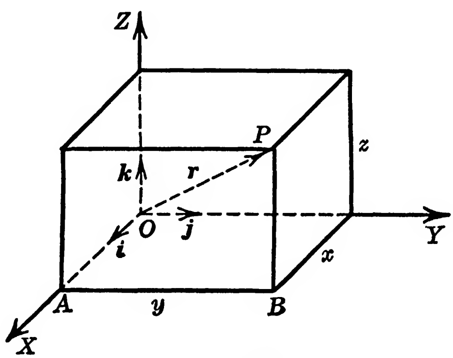

FIGURE 208.

Since OBP is a right triangle, the distance r from the origin to $P(x, y, z)$ is determined by the equation

$$r^2 = \overline{OB}^2 + \overline{BP}^2 = x^2 + y^2 + z^2,$$

whence

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{3}$$

Similarly,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{4}$$

is a vector with components A_x, A_y, A_z along the axes. These components are edges of a box having the vector as diagonal. The length of $\mathbf{A}$ is therefore

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{5}$$

In particular, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ has the components $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ (Figure 209). Thus

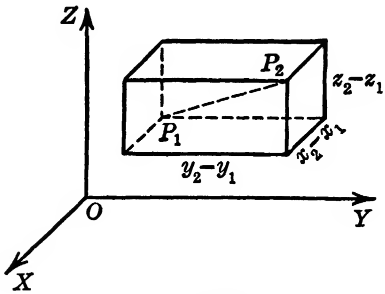

FIGURE 209.

$$\overrightarrow{P_1P_2} = \mathbf{i}(x_2 - x_1) + \mathbf{j}(y_2 - y_1) + \mathbf{k}(z_2 - z_1), \tag{6}$$

and the distance between these points is

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{7}$$

Example. Given $A(0, -1, 4)$, $B(2, 2, -2)$, find the vector $\overrightarrow{AB}$ and the distance AB.

By equation (6) the vector is

$$\overrightarrow{AB} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}.$$

The distance between the points is

$$AB = \sqrt{2^2 + 3^2 + 6^2} = 7.$$

166. The Scalar Product of Two Vectors. The *scalar*, or *dot*, *product* of two vectors $\mathbf{A}$ and $\mathbf{B}$ (indicated by placing a dot between the factors) is defined as the scalar

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta, \tag{1}$$

where θ is the angle between the two vectors.

The quantity

$$b = |\mathbf{B}| \cos\theta$$

is the component of $\mathbf{B}$ along $\mathbf{A}$. *The scalar product*

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta = |\mathbf{A}|\, b$$

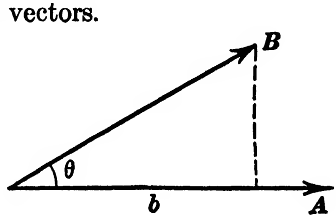

FIGURE 210.

is thus equal to the product of $|\mathbf{A}|$ *and the component of* $\mathbf{B}$ *along* $\mathbf{A}$.

It is also equal to the product of $|\mathbf{B}|$ and the component of **A** along **B**.

From the definition it is evident that

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}, \tag{2}$$

that is, the scalar product is commutative (§105).

If b, c are the components of **B**, **C** along **A**, it is clear from a diagram (Figure 211) that $b + c$ is the component of $\mathbf{B} + \mathbf{C}$ along **A**. The equation

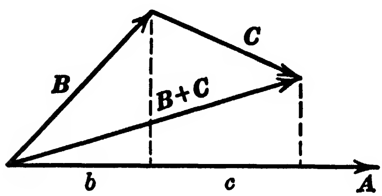

FIGURE 211.

$$|\mathbf{A}|\,(b + c) = |\mathbf{A}|\,b + |\mathbf{A}|\,c$$

is thus equivalent to

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}. \tag{3}$$

Since the product is commutative, this requires also

$$(\mathbf{B} + \mathbf{C})\cdot\mathbf{A} = \mathbf{B}\cdot\mathbf{A} + \mathbf{C}\cdot\mathbf{A}. \tag{4}$$

Equations (3) and (4) express the distributive law (§106) for scalar multiplication of vectors.

If **A** and **B** are perpendicular, $\cos\theta$ is zero and

$$\mathbf{A}\cdot\mathbf{B} = 0.$$

Conversely, *if the scalar product is zero, one of the factors is zero or else the two are perpendicular.*

Since the product may be zero when neither factor is zero, it follows that *division by a vector cannot be performed.* Thus, if **A** is not zero and

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C},$$

we cannot conclude that

$$\mathbf{B} = \mathbf{C}$$

but merely that

$$\mathbf{A}\cdot(\mathbf{B} - \mathbf{C}) = 0,$$

and hence $\mathbf{B} - \mathbf{C}$ is zero or perpendicular to **A**.

If **A** and **B** have the same direction, $\cos\theta = 1$, and $\mathbf{A}\cdot\mathbf{B}$ is equal to the product of the lengths of the two vectors. In particular,

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 \tag{5}$$

is the square of the length of **A**. The scalar product of a vector

with itself is sometimes written as the square of the vector. Thus

$$\mathbf{A}\cdot\mathbf{A} = \mathbf{A}^2, \tag{6}$$

but no other powers are defined.

Applying the definition of scalar product to the unit vectors **i**, **j**, **k**, we obtain

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \tag{7}$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

If

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k},$$

by expanding and using (7) we find

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{8}$$

Example 1. Show that the vectors

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k},$$

$$\mathbf{B} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

are perpendicular.

The scalar product is

$$\mathbf{A}\cdot\mathbf{B} = 6 - 2 - 4 = 0.$$

Hence the vectors are perpendicular.

Example 2. Find the component of

$$\mathbf{A} = 4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

along

$$\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

If θ is the angle between **A** and **B**, the component of **A** along **B** is

$$|\mathbf{A}|\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|} = \frac{2}{3}.$$

Example 3. The vector

$$A = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

is expressed as the sum of two vectors **A**′ and **A**″ respectively along and perpendicular to

$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find **A**′ and **A**″.

The projection of **A** on **B** is

$$\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|}.$$

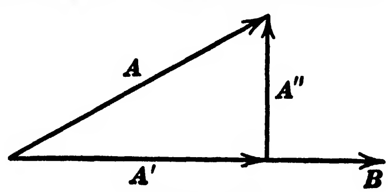

FIGURE 212.

The vector $\mathbf{A}'$ is the product of this projection and the unit vector

$$\frac{\mathbf{B}}{|\mathbf{B}|}.$$

Thus

$$\mathbf{A}' = \frac{(\mathbf{A}\cdot\mathbf{B})\mathbf{B}}{\mathbf{B}^2} = \frac{1}{2}(2\mathbf{i} - \mathbf{j} - \mathbf{k}),$$

$$\mathbf{A}'' = \mathbf{A} - \mathbf{A}' = \tfrac{3}{2}(\mathbf{j} - \mathbf{k}).$$

167. Work. The simplest illustration of the scalar product is furnished by mechanical work. When the point of application of a force $\mathbf{F}$ experiences a displacement which in magnitude and direction is represented by the vector $\mathbf{s}$ (Figure 213), the work

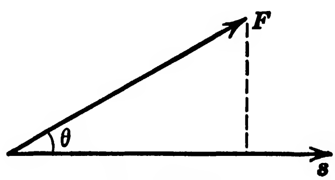

FIGURE 213.

done by the force is defined as the product of the displacement and the component of the force along the direction of the displacement. That is,

$$\text{Work} = |\mathbf{s}|\,|\mathbf{F}|\cos\theta = \mathbf{F}\cdot\mathbf{s}, \tag{1}$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{s}$.

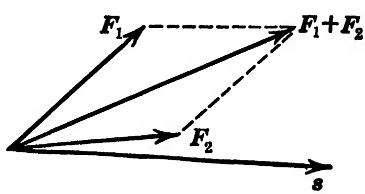

FIGURE 214.

If $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ is the resultant of two forces $\mathbf{F}_1$, $\mathbf{F}_2$ acting at the same point, the distributive law

$$\mathbf{F}\cdot\mathbf{s} = (\mathbf{F}_1 + \mathbf{F}_2)\cdot\mathbf{s} = \mathbf{F}_1\cdot\mathbf{s} + \mathbf{F}_2\cdot\mathbf{s} \tag{2}$$

states that the work of the resultant is equal to the sum of the works done by the separate forces.

When the point of application of a force $\mathbf{F}$ experiences two separate displacements $\mathbf{s}_1 = \overrightarrow{AB}$, $\mathbf{s}_2 = \overrightarrow{BC}$ (Figure 215), the distributive law

$$\mathbf{F}\cdot\mathbf{s}_1 + \mathbf{F}\cdot\mathbf{s}_2 = \mathbf{F}\cdot(\mathbf{s}_1 + \mathbf{s}_2)$$

shows that the sum of the works in the two displacements is equal to the work done by $\mathbf{F}$ in the single displacement

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$$

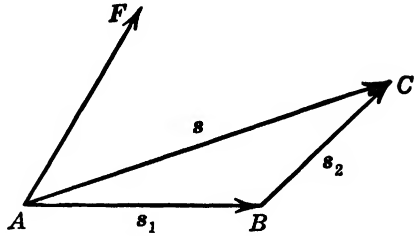

FIGURE 215.

from A to C. In a similar way we show that the total work done by a constant force (such as the force of gravity on a body) when its point of application experiences any number of consecutive displacements is equal to the work done by the force in a single displacement from the beginning to the end of motion. In particular, the total work of a constant force is zero when its point of application traverses consecutively the sides of a closed polygon.

Variable forces will be discussed later (§200).

168. Angle between Two Lines. By the angle between two lines which do not intersect we mean the angle between parallels to these lines which do intersect.

The angles a directed line makes with the coördinate axes OX, OY, OZ are usually represented by the letters α, β, γ, and the cosines of these angles

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma \tag{1}$$

are called the *direction cosines* of the line.

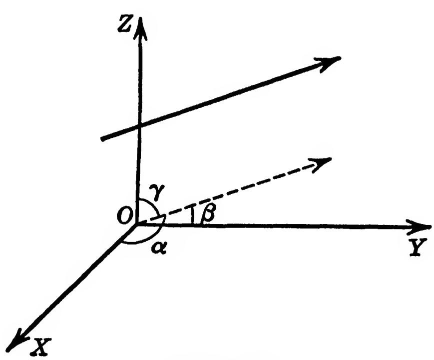

FIGURE 216.

Since the components of a vector

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$$

are its projections on the coördinate axes, the direction cosines of the vector are evidently

$$\cos\alpha = \frac{V_x}{|\mathbf{V}|} = \frac{V_x}{\sqrt{V_x^2 + V_y^2 + V_z^2}},$$

$$\cos\beta = \frac{V_y}{|\mathbf{V}|} = \frac{V_y}{\sqrt{V_x^2 + V_y^2 + V_z^2}},$$

$$\cos\gamma = \frac{V_z}{|\mathbf{V}|} = \frac{V_z}{\sqrt{V_x^2 + V_y^2 + V_z^2}}.$$

In particular, a vector of unit length has components equal to its direction cosines. Thus

$$\mathbf{u} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma \tag{2}$$

is the unit vector along the line with direction angles α, β, γ.

If two directed lines make with the coördinate axes the angles α_1, β_1, γ_1 and α_2, β_2, γ_2, the unit vectors along those lines are

$$\mathbf{u}_1 = \mathbf{i}\cos\alpha_1 + \mathbf{j}\cos\beta_1 + \mathbf{k}\cos\gamma_1,$$

$$\mathbf{u}_2 = \mathbf{i}\cos\alpha_2 + \mathbf{j}\cos\beta_2 + \mathbf{k}\cos\gamma_2.$$

The angle θ between the lines is therefore determined by the equation

$$\cos\theta = \mathbf{u}_1\cdot\mathbf{u}_2 = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \tag{3}$$

If the two directions coincide, $\theta = 0$, and we have

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma \tag{4}$$

as a relation satisfied by the direction cosines of any line. This is the space analogue of the equation $\cos^2\theta + \sin^2\theta = 1$ in the plane.

From the equation

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta$$

we get

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} \tag{5}$$

as the cosine of the angle between any two vectors.

Example. The lines AB, AC in Figure 217 make with the z-axis the angles α, β respectively. Find the angle between the two lines.

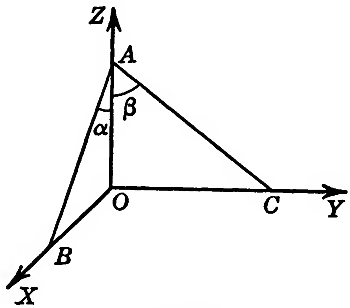

FIGURE 217.

The unit vectors along $\overrightarrow{AB}$, $\overrightarrow{AC}$ are

$$\mathbf{u}_1 = \mathbf{i} \sin \alpha - \mathbf{k} \cos \alpha,$$

$$\mathbf{u}_2 = \mathbf{j} \sin \beta - \mathbf{k} \cos \beta.$$

The angle θ between the two is therefore given by the equation

$$\cos \theta = \mathbf{u}_1 \cdot \mathbf{u}_2 = \cos \alpha \cos \beta.$$

169. Rotation of Axes. As an illustration of direction cosines we take the transformation to new axes having the same origin.

Let O-XYZ and O-$X'Y'Z'$ be two sets of rectangular axes, the unit vectors along the former being $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, those along the latter $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$. The axes of one set determine nine angles with those of the other set. Let the numbers in the following table be the cosines

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

of those angles, the number at the intersection of any row and column being the cosine of the angle between the axes indicated at the end of the row and the top of the column. These cosines are the scalar products of the corresponding unit vectors. Thus

$$l_3 = \mathbf{i} \cdot \mathbf{k}' \tag{1}$$

is the cosine of the angle between OX and OZ'.

If x, y, z and x', y', z' are the coördinates of the same point P with respect to $O\text{-}XYZ$ and $O\text{-}X'Y'Z'$, the vector $\overrightarrow{OP}$ is represented by two equivalent expressions

$$\mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z'. \tag{2}$$

This is the vector equation for transformation of coördinates.

To obtain any coördinate of one system in terms of those of the other system we multiply both sides of (2) by the corresponding unit vector and use the appropriate equations of the form (1). Thus, multiplying by **i**, **j**, **k** respectively, we get

$$\begin{aligned} x &= l_1x' + l_2y' + l_3z', \\ y &= m_1x' + m_2y' + m_3z', \\ z &= n_1x' + n_2y' + n_3z'. \end{aligned} \tag{3}$$

Example. The coördinate axes are rotated about the origin to new positions OX', OY', OZ' along which the unit vectors are

$$\begin{aligned} i' &= \tfrac{1}{9}[i + 4j - 8k], \\ j' &= \tfrac{1}{9}[4i + 7j + 4k], \\ k' &= \tfrac{1}{9}[8i - 4j - k]. \end{aligned}$$

Determine the new coördinates of the point $P(x, y, z)$.

The vector transformation is

$$i'x' + j'y' + k'z' = ix + jy + kz.$$

Multiplying the left side of this equation by i', j', k' and the right side by the equivalent values given above, we get

$$\begin{aligned} x' &= \tfrac{1}{9}[x + 4y - 8z], \\ y' &= \tfrac{1}{9}[4x + 7y + 4z], \\ z' &= \tfrac{1}{9}[8x - 4y - z]. \end{aligned}$$

170. The Vector Product. The *vector*, or *cross*, *product* of two vectors **A** and **B** (indicated by placing a small cross between the two factors) is defined as the vector

$$\mathbf{V} = \mathbf{A} \times \mathbf{B} \tag{1}$$

perpendicular to both **A** and **B**, with magnitude

$$|\mathbf{V}| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta, \tag{2}$$

and so directed that right-handed rotation about **V** through an angle θ of not more than 180° carries **A** into **B**.

Since

$$h = |\mathbf{B}| \sin \theta$$

is the altitude of the parallelogram with sides **A** and **B**, the product $\mathbf{A} \times \mathbf{B}$ is equal in magnitude to the area of that parallelogram.

The product $\mathbf{B} \times \mathbf{A}$ has the same magnitude as $\mathbf{A} \times \mathbf{B}$, but the rotation which carries **B** into **A** is opposite to that which carries **A** into **B**. Hence

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \qquad (3)$$

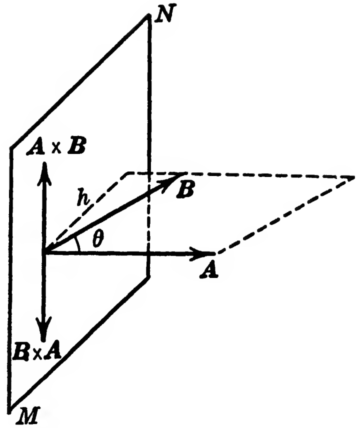

FIGURE 218.

Thus *the vector product is not commutative.*

If **A** and **B** are parallel, θ is zero or 180° and

$$\mathbf{A} \times \mathbf{B} = 0.$$

Conversely, *if the vector product is zero, one of the vectors is zero or else the two are parallel.*

The product $\mathbf{A} \times \mathbf{B}$ can be considered as the result obtained by projecting **B** on the plane MN perpendicular to **A**, rotating the projection 90° in the positive direction about **A**, and multiplying the resulting vector by $|\mathbf{A}|$. Each of these operations changes a closed polygon into a closed polygon. If they are performed on the sides **B**, **C**, $\mathbf{B} + \mathbf{C}$ of a triangle, the resulting vectors

$$\mathbf{A} \times \mathbf{B}, \quad \mathbf{A} \times \mathbf{C}, \quad \mathbf{A} \times (\mathbf{B} + \mathbf{C})$$

thus form the sides of a second triangle. Therefore

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \quad (4)$$

whence

$$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A}, \quad (5)$$

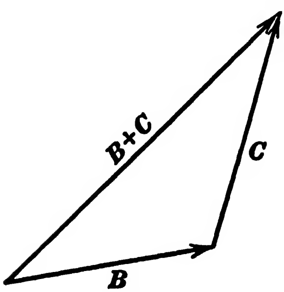

FIGURE 219.

since each term of (5) merely differs in algebraic sign from the corresponding term of (4). These equations express the distributive law of vector multiplication.

In case of the unit vectors **i**, **j**, **k** the definition of vector product gives immediately

$$\begin{aligned}
\mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \\
\mathbf{i} \times \mathbf{j} &= \mathbf{k} = -\mathbf{j} \times \mathbf{i}, \\
\mathbf{j} \times \mathbf{k} &= \mathbf{i} = -\mathbf{k} \times \mathbf{j}, \\
\mathbf{k} \times \mathbf{i} &= \mathbf{j} = -\mathbf{i} \times \mathbf{k}.
\end{aligned} \tag{6}$$

By means of these relations the cross product of two vectors can be expressed in terms of their components. Thus, if

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k},$$

by expanding and using (6) we find

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}. \tag{7}$$

As a determinant this can be written

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{8}$$

Example 1. Find a vector perpendicular to the plane through $A(0, 2, 2)$, $B(2, 0, -1)$, $C(3, 4, 0)$.

The vectors

$$\overrightarrow{AB} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k},$$

$$\overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

are in the plane, and consequently their product

$$\overrightarrow{AB} \times \overrightarrow{AC} = 10\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$$

is perpendicular to the plane. Since division by 5 does not alter the direction, we may take

$$\mathbf{N} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

as the required vector.

Example 2. Find the area of the triangle formed by the points A, B, C in the preceding problem.

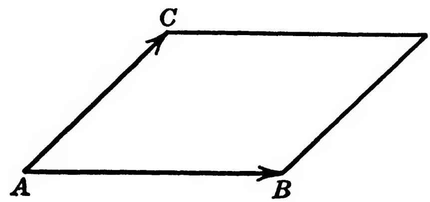

FIGURE 220.

The product

$$\overrightarrow{AB} \times \overrightarrow{AC} = 10\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$$

has a magnitude equal to the area of the parallelogram with vertices A, B, C. The area of the triangle is thus

$$\tfrac{1}{2}\sqrt{10^2 + 5^2 + 10^2} = \tfrac{15}{2}.$$

Example 3. Find the shortest distance between the line through $A(0, 1, 2)$, $B(1, 2, 2)$ and that through $C(2, 0, 1)$, $D(2, 1, 2)$.

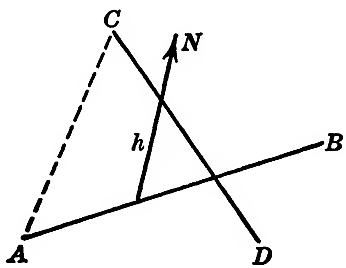

FIGURE 221.

The shortest distance is the common perpendicular to the two lines. To obtain this we determine a vector

$$\mathbf{N} = \overrightarrow{AB} \times \overrightarrow{CD} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

perpendicular to both lines and project on this a vector, such as

$$\overrightarrow{AC} = 2\mathbf{i} - \mathbf{j} - \mathbf{k},$$

which extends from a point on one line to a point on the other. The shortest distance is thus

$$h = \frac{\overrightarrow{AC} \cdot \mathbf{N}}{|\mathbf{N}|} = \frac{2}{\sqrt{3}}.$$

171. Moment of a Force. If $\mathbf{F}$ is a force applied to a body at the point P, and $\mathbf{r}$ is the vector from O to P, the product

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{1}$$

is called *the moment of* $\mathbf{F}$ *about the point* O.

The force tends to rotate the body about an axis through O perpendicular to the plane of O and $\mathbf{F}$. The moment is a vector extending along that axis in the direction a right-handed screw advances under such a rotation. In magnitude it is equal to the product of the force by its lever arm, or perpendicular distance from the axis.

Moment is usually defined with respect to an axis and not with respect to a point, the moment of $\mathbf{F}$ about OZ being defined as

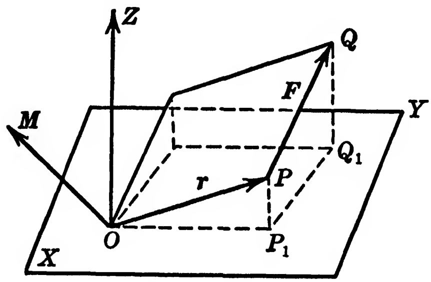

FIGURE 222.

the product of its projection P_1Q_1 on the plane OXY perpendicular to OZ, by the distance from O to the line P_1Q_1 (§134). But that, being the area of the parallelogram OP_1Q_1, is the projection of OPQ on XY. Now the parallelogram OPQ has an area equal to $|\mathbf{M}|$ and makes with XY an angle equal to that between $\mathbf{M}$ and OZ. Thus the projection of the parallelogram on XY is equal to the projection of $\mathbf{M}$ on OZ. Hence, *the moment of* $\mathbf{F}$ *about any axis through* O *is equal to the component of* $\mathbf{M}$ *along that axis.*

The distributive formulas

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2, \tag{2}$$

$$(\mathbf{r}_1 + \mathbf{r}_2) \times \mathbf{F} = \mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times \mathbf{F}, \tag{3}$$

state that the moment of the resultant of two forces is equal to the sum of their moments, and that the moment of $\mathbf{F}$ about O_2 can be found by taking the moment about O_1 and adding the moment about O_2 of a force $\mathbf{F}$ at O_1. These formulas are to a con-

siderable extent responsible for the importance of moment and the vector product in mechanics.

Example 1. The force

$$\mathbf{F} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

is applied at the point $P(2, -1, 3)$. Find its moment about the point $A(1, 2, 3)$.

The required moment is

$$\mathbf{M} = \overrightarrow{AP} \times \mathbf{F} = (\mathbf{i} - 3\mathbf{j}) \times (3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}.$$

The components 3, 1, 7 are the moments of $\mathbf{F}$ about axes through A parallel to OX, OY, OZ.

Example 2. Given $A(1, 2, 3)$, $B(3, 1, 1)$, find the moment of the force $\mathbf{F}$ in the preceding problem about the axis $\overrightarrow{AB}$.

The moment of $\mathbf{F}$ about the axis $\overrightarrow{AB}$ is the component of

$$\mathbf{M} = \overrightarrow{AP} \times \mathbf{F} = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}$$

along the vector

$$\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}.$$

The value required is therefore

$$\frac{\mathbf{M}\cdot\overrightarrow{AB}}{|\overrightarrow{AB}|} = -3.$$

The negative sign signifies that $\mathbf{F}$ tends to produce rotation in the negative direction about $\overrightarrow{AB}$.

172. Products of Three or More Vectors. By scalar and vector multiplication of three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, triple products of the following types are obtained:

$$(1) \quad (\mathbf{A}\cdot\mathbf{B})\mathbf{C},$$

$$(2) \quad \mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}),$$

$$(3) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

In these expressions parentheses have been introduced to show the order in which the multiplications are performed. Thus to obtain

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

we first obtain $\mathbf{B} \times \mathbf{C}$ and then multiply by $\mathbf{A}$. In general the vector $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ does not have the same value. Since the product $\mathbf{BC}$

without a dot or cross has not been defined, the expression $\mathbf{A}\cdot\mathbf{BC}$ can only mean $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$. Also, since both factors of a cross product must be vectors, $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ can only mean $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. In products of types (1) and (2) the parentheses may therefore be omitted. For added clearness, however, they are often inserted.

The product $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ is merely $\mathbf{A}\cdot\mathbf{B}$ times the vector $\mathbf{C}$. It is thus a vector $\mathbf{A}\cdot\mathbf{B}$ times as long as $\mathbf{C}$ and having the same direction if $\mathbf{A}\cdot\mathbf{B}$ is positive, the opposite direction if $\mathbf{A}\cdot\mathbf{B}$ is negative.

The product

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$$

is a scalar equal in magnitude to the volume of the parallelepiped with edges $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. This is easily seen since $\mathbf{B}\times\mathbf{C}$ is a vector perpendicular to the base, of magnitude equal to the area of the base, and

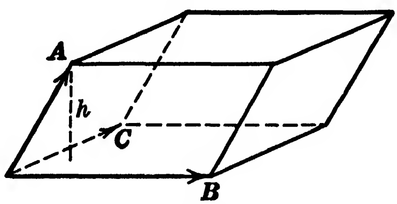

FIGURE 223.

the projection of $\mathbf{A}$ on this vector is the altitude h of the parallelepiped. Hence

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = h\,|\mathbf{B}\times\mathbf{C}|$$

is equal to the volume.

Inspection of the diagram shows that

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}. \tag{1}$$

The product is thus not changed by a cyclical permutation of the letters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. The other three products

$$\mathbf{A}\cdot\mathbf{C}\times\mathbf{B} = \mathbf{B}\cdot\mathbf{A}\times\mathbf{C} = \mathbf{C}\cdot\mathbf{B}\times\mathbf{A} \tag{2}$$

are easily seen to have the opposite algebraic sign.

Since

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}),$$

equation (1) shows that

$$\mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C}. \tag{3}$$

In the triple scalar product the dot and cross can therefore be interchanged without changing the value of the result.

If

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$
$$\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k},$$

the identity

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{4}$$

may be proved by showing that the product and determinant are formed by the same laws from the components of **A**, **B**, **C**.

The product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

is a vector sometimes encountered in work involving repeated multiplication. It has no very simple geometrical interpretation but can be reduced to simpler form by means of the equation

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}. \tag{5}$$

This and the similar equation

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A} \tag{6}$$

can be proved by expressing **A**, **B**, **C** in terms of **i**, **j**, **k** and expanding both sides.

By means of the above identities products involving four or more vectors can be reduced to a form in which each term is at most a scalar factor times the cross product of two vectors.

Example. Expand $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$. This can be considered as the scalar triple product of $\mathbf{A}\times\mathbf{B}$, **C**, and **D**. By the interchange of dot and cross we thus have

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= [(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\cdot\mathbf{D} \\ &= [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}]\cdot\mathbf{D} \\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D}).\end{aligned}$$

173. Space Loci. The locus of points in space with coördinates x, y, z which satisfy an equation

$$f(x, y, z) = 0$$

is in general a surface. For example,

$$z = 0$$

represents the xy-plane, and

$$x^2 + y^2 + z^2 = 1$$

is the locus of points at distance 1 from the origin, that is, the surface of a sphere with center at the origin.

In particular cases real solutions may, however, occur only at discrete points or at points on a line or curve. Thus the only real points for which

$$x^2 + y^2 = 0$$

are the points $x = 0$, $y = 0$ on the z-axis, and

$$x^2 + y^2 + z^2 = 0$$

is satisfied only at the origin.

If an equation contains only two rectangular coördinates, the surface it represents is not changed by motion parallel to the axis of the missing coördinate. Such an equation thus represents a cylindrical surface, that is, a surface generated by a line parallel to a fixed direction and cutting a fixed curve. The equation

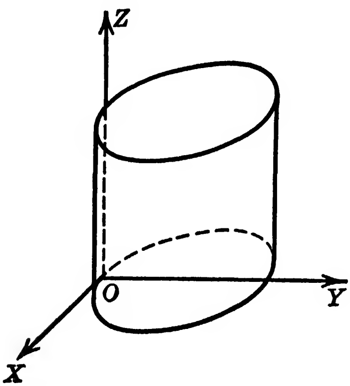

FIGURE 224.

$$x^2 + y^2 = 2y,$$

for example, represents a circle in the xy-plane. In space the equation is satisfied by the coördinates of all points vertically above or below this circle. The locus is therefore the cylinder with axis parallel to OZ and intersecting the xy-plane in the above circle.

Two simultaneous equations

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0$$

usually represent a curve or straight line; for each equation represents a surface, and hence two simultaneous equations represent the points common to two surfaces. Thus the equations

$$x^2 + y^2 + z^2 = 2, \quad z = 1$$

represent the circle in which a sphere and plane intersect. In particular cases, the equations may, however, be satisfied only at discrete points, or may have a common factor and the locus include a surface.

Three simultaneous equations in x, y, z are usually satisfied only by the coördinates of a definite number of points, which may be obtained by solving the equations simultaneously. In particular cases, there may be no solution or the equations may be satisfied by the coördinates of all points on a curve or surface.

Example 1. Show that

$$x^2 + y^2 + z^2 - 2x + 4y + 2z + 2 = 0$$

is the equation of a sphere, and find its center and radius.

Completing the squares of terms in x, y, z separately, we get

$$(x - 1)^2 + (y + 2)^2 + (z + 1)^2 = 4.$$

Thus the point (x, y, z) is at distance 2 from the point $(1, -2, -1)$. The locus is therefore a sphere of radius 2 and center $(1, -2, -1)$.

Example 2. Determine the locus represented by

$$x + z = 1.$$

In the xz-plane the locus is the line AB (Figure 225). In space the

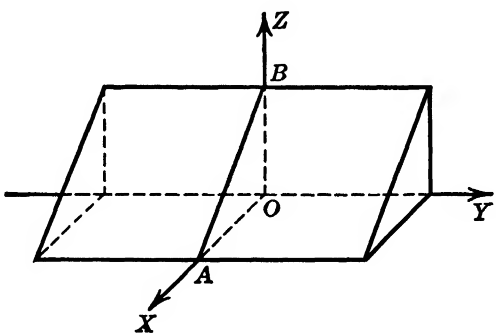

FIGURE 225.

locus is the plane generated by lines which are parallel to the y-axis and cut AB.

Example 3. The equations

$$x^2 + y^2 + z^2 = 1, \quad x + y + z = 0 \tag{1}$$

represent a space curve. Find the projection of this curve on the xy-plane.

To do this we eliminate z from the equations of the curve. The resulting equation

$$x^2 + y^2 + (x + y)^2 = 1 \tag{2}$$

represents a cylinder with generators parallel to OZ. Since this equation is obtained by combining the equations of the curve, it is satisfied by the

coördinates of any point on the curve. The cylinder thus passes through the curve, and its intersection with the xy-plane is consequently the projection required. Considered as an equation in the xy-plane, (2) is the equation of the intersection and therefore the equation of the projection.

174. Equation of a Plane. Let $P_1(x_1, y_1, z_1)$ be a fixed point in a given plane and

$$\overrightarrow{MN} = \mathbf{i}A + \mathbf{j}B + \mathbf{k}C \tag{1}$$

a vector perpendicular to the plane. A point $P(x, y, z)$ will lie in the given plane if and only if

$$\overrightarrow{P_1P} = \mathbf{i}(x - x_1) + \mathbf{j}(y - y_1) + \mathbf{k}(z - z_1) \tag{2}$$

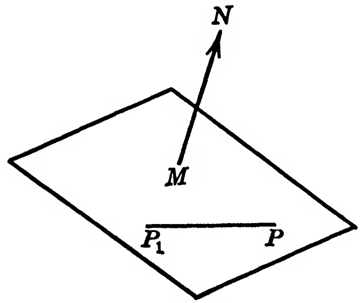

FIGURE 226.

is perpendicular to $\overrightarrow{MN}$. By taking the scalar product of (1) and (2) we thus obtain

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \tag{3}$$

as the equation of the plane which passes through (x_1, y_1, z_1) and is perpendicular to the vector with components A, B, C.

A line perpendicular to a plane at a point is called *normal* to the plane at that point. If α, β, γ are the angles such a normal makes with the coördinate axes,

$$\mathbf{i} \cos \alpha + \mathbf{j} \cos \beta + \mathbf{k} \cos \gamma \tag{4}$$

is a unit vector along the normal. Using this instead of (1), we get

$$(x - x_1) \cos \alpha + (y - y_1) \cos \beta + (z - z_1) \cos \gamma = 0 \tag{5}$$

as the equation of the plane which passes through (x_1, y_1, z_1) and has a normal with direction angles α, β, γ.

Equations (3) and (5) are of first degree in x, y, z. Any plane, therefore, has an equation of first degree in rectangular coördinates.

Conversely, any equation of first degree in rectangular coördinates represents a plane. For any such equation has the form

$$Ax + By + Cz + D = 0, \tag{6}$$

A, B, C, D being constants. If x_1, y_1, z_1 satisfy this equation,

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Subtraction gives

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

which has the form (3). Thus (6) represents a plane perpendicular to the vector with components A, B, C.

Example 1. Find the angle between the planes

$$x + 2y + 2z = 0, \quad x - 4y + 8z = 5.$$

The vectors with components 1, 2, 2 and 1, -4, 8 are normal to the planes. The angle between two planes is the same as that between their normals. By equation (5), §168, we thus have

$$\cos\theta = \frac{1 - 8 + 16}{\sqrt{9}\ \sqrt{81}} = \frac{1}{3}$$

as the cosine of the required angle. The planes determine two angles less than 180°, the smaller being $\cos^{-1}\frac{1}{3}$.

Example 2. Find the equation of the plane which passes through $P_1(1, -1, 2)$, $P_2(3, 2, 1)$, and $P_3(2, 1, 3)$.

The equation of the plane can be written

$$Ax + By + Cz + D = 0.$$

Since this is satisfied by the coördinates of P_1, P_2, P_3, we have

$$A - B + 2C + D = 0,$$

$$3A + 2B + C + D = 0,$$

$$2A + B + 3C + D = 0.$$

The four equations being homogeneous in A, B, C, D, the condition

$$\begin{vmatrix} x & y & z & 1 \\ 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 1 \\ 2 & 1 & 3 & 1 \end{vmatrix} = 0$$

must be satisfied (§75). By expanding the determinant we get

$$5x - 3y + z = 10$$

as the equation of the plane.

Another method of obtaining the same result is to use the fact that the vector

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

is normal to the plane. Since the plane passes through $P_1(1, -1, 2)$ its equation is

$$5(x - 1) - 3(y + 1) + z - 2 = 0.$$

175. Equations of a Straight Line. A straight line is the intersection of two planes. In rectangular coördinates it is therefore represented by two simultaneous equations of the first degree. Thus

$$x + y + z = 6,$$

$$2x - 3y + 2z = 2$$

are equations of a straight line. Since there are infinitely many pairs of planes through the line it is represented by infinitely many pairs of first-degree equations.

To obtain equations of more definite form choose a positive direction along the line and let α, β, γ be the angles this direction makes with the coördinate axes. If $P_1(x_1, y_1, z_1)$ is a fixed point on the line and $P(x, y, z)$ a variable point, the vector

$$\overrightarrow{P_1P} = \mathbf{i}(x - x_1) + \mathbf{j}(y - y_1) + \mathbf{k}(z - z_1) \tag{1}$$

is then parallel to the vector

$$\mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma. \tag{2}$$

Thus

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma} \tag{3}$$

are equations of the line which passes through (x_1, y_1, z_1) and makes with the coördinate axes the angles α, β, γ.

Instead of (2) we can use any vector

$$\mathbf{i}A + \mathbf{j}B + \mathbf{k}C \tag{4}$$

parallel to the line. We then get

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C} \tag{5}$$

as equations of the line which passes through (x_1, y_1, z_1) and is parallel to the vector with components A, B, C.

Example 1. Find the equations of the line which passes through the points $P_1(-2, 1, 0)$ and $P_2(1, 0, 1)$.

The line passes through (1, 0, 1) and is parallel to the vector

$$\overrightarrow{P_1P_2} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Its equations are therefore

$$\frac{x-1}{3} = \frac{y}{-1} = \frac{z-1}{1}.$$

Example 2. Find direction cosines of the line represented by

$$x + y + z = 6,$$

$$2x - 3y + 2z = 2.$$

The vectors

$$\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

$$\mathbf{n}_2 = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

are perpendicular to the two planes. Their product

$$\mathbf{n}_1 \times \mathbf{n}_2 = 5\mathbf{i} - 5\mathbf{k}$$

is therefore a vector parallel to the line. The direction cosines of this vector are

$$\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}.$$

If the positive direction along the line is taken as that of the vector, these are also the direction cosines of the line. If the line has the opposite direction, its direction cosines are

$$-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}.$$

176. Cylindrical Coördinates. Any three numbers which determine the position of a point in space can be used to represent it.

If M is the projection of a point P upon the xy-plane and r, θ are the polar coördinates of M in the xy-plane, the numbers r, θ, z are called the cylindrical coördinates of P. The name is suggested by the fact that the locus of points

$$r = \text{constant}$$

is a cylinder.

The angle θ is measured from the x-axis and is considered positive when its direction is that which carries OX into OY by rotation through 90°. The coördinate r is positive when M is on the terminal side of θ.

The equations connecting the rectangular and cylindrical coördinates of the same point can be read from a diagram (Figure 227). The most important of these are

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}. \tag{1}$$

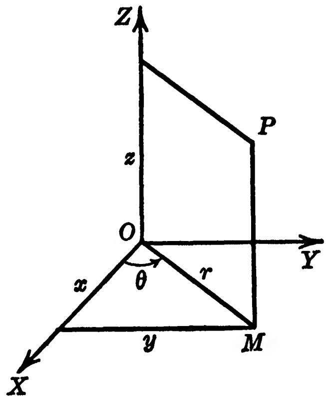

FIGURE 227.

Example. Find the equation of a right circular cone with vertex at the origin, axis OZ, and vertical angle 2α.

The cylindrical coördinates of any point P on the surface of the cone are

$$MP = r, \quad OM = z.$$

Hence

$$r = z\tan\alpha$$

is the equation of the cone.

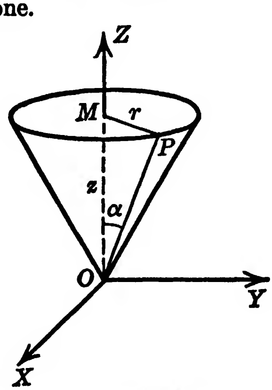

FIGURE 228.

177. Surface of Revolution. The surface generated by revolving a plane curve about an axis in its plane is called a *surface of revolution.*

Suppose, for example, that the curve

$$f(x, z) = 0$$

in the xz-plane is rotated about the z-axis. Any point P on the resulting surface has cylindrical coördinates r, z equal to the coördinates x, z at some point Q on the curve. These cylindrical coördinates thus satisfy the equation

$$f(r, z) = 0,$$

which is therefore the equation of the surface generated.

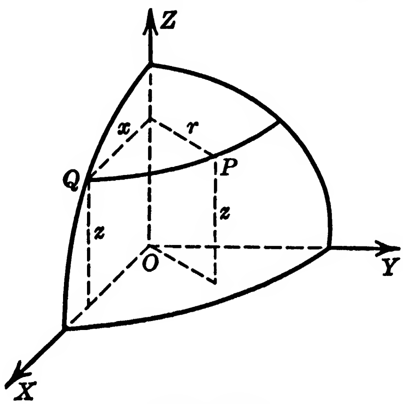

FIGURE 229.

Since $r = \sqrt{x^2 + y^2}$, the corresponding Cartesian equation is

$$f(\sqrt{x^2 + y^2}, z) = 0.$$

Similar equations are obtained for the surfaces generated by rotating about OX or OY.

Example 1. Find the equation of the cone generated by rotating the line

$$x + y = 1$$

about the x-axis.

The equation is obtained by leaving x unchanged and replacing y by $\sqrt{y^2 + z^2}$. Thus

$$x \pm \sqrt{y^2 + z^2} = 1,$$

and consequently

$$y^2 + z^2 = (1 - x)^2$$

is the equation required.

Example 2. The circle

$$x^2 + z^2 = 2ax$$

is rotated about the z-axis. In cylindrical coördinates find the equation of the surface generated.

The equation is obtained by replacing x by r. Thus

$$r^2 + z^2 = 2ar$$

is the equation of the surface.

178. Quadric Surfaces. The locus of an equation of the second degree in x, y, z is called a *quadric surface.* All plane sections of such a surface are curves represented by Cartesian equations of the second degree, that is, conic sections (§65). The appearance of the surface can be determined by the sections in a suitably chosen set of planes.

The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

The xz- and yz-planes cut this surface in ellipses

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

having a common vertical axis. The section in a horizontal plane $z = k$ is an ellipse

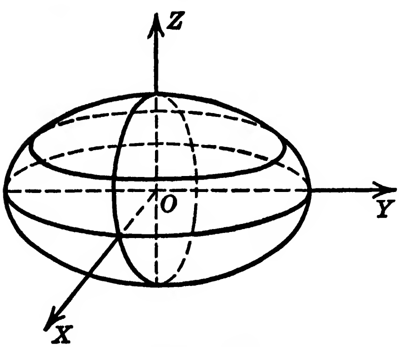

FIGURE 230.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$$

the axes of which are chords of the two vertical ellipses.

The distances a, b, c, called *semiaxes,* are the intercepts on the coördinate axes. If two of the semiaxes are equal, the ellipsoid is called a *spheroid.* It is then a surface of revolution obtained by revolving an ellipse about one of its axes. If all the semiaxes are equal, the ellipsoid is a *sphere.*

The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{2}$$

The sections in the xz- and yz-planes are hyperbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

having a common vertical axis. The section in a horizontal plane $z = k$ is an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2},$$

with the ends of its axes on the two hyperbolas.

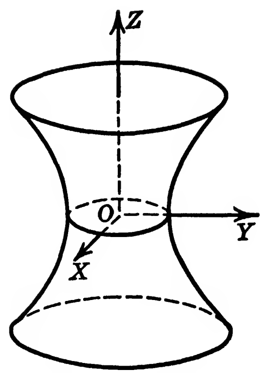

FIGURE 231.

For different values of k all these ellipses are similar, the smallest being in the xy-plane.

If $a = b$ the surface is a hyperboloid of revolution generated by revolving a hyperbola about its conjugate axis.

The hyperboloid of two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{3}$$

The section in a plane $x = k$ is imaginary if $|x| < a$. The surface therefore consists of two parts, one for values $x > a$, the other for values $x < -a$. The surface cuts the xy- and xz-planes in hyperbolas, and is generated by ellipses with axes terminating on these hyperbolas.

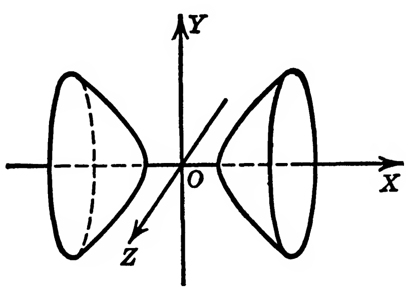

FIGURE 232.

If $b = c$, the surface is a hyperboloid of revolution generated by revolving a hyperbola about its transverse axis.

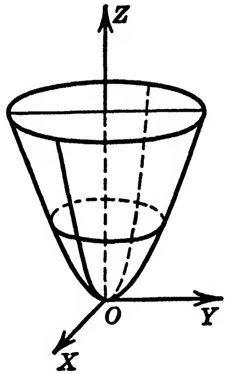

FIGURE 233.

The elliptic paraboloid

$$z = ax^2 + by^2, \tag{4}$$

a and b having the same algebraic sign.

Sections in the xz- and yz-planes are parabolas

$$z = ax^2, \quad z = by^2.$$

The section in a horizontal plane $z = k$ is an ellipse

$$ax^2 + by^2 = k$$

if k has the same sign as a and b, and is imaginary if k has the opposite sign. The surface is thus generated by ellipses, the axes of which are chords of the vertical parabolas.

If $a = b$, the surface is a paraboloid of revolution generated by revolving a parabola about its axis.

The hyperbolic paraboloid

$$z = by^2 - ax^2,$$

a and b being positive.

The section in the zx-plane is a parabola $z = -ax^2$ with axis extending downward; that in the yz-plane, a parabola with axis

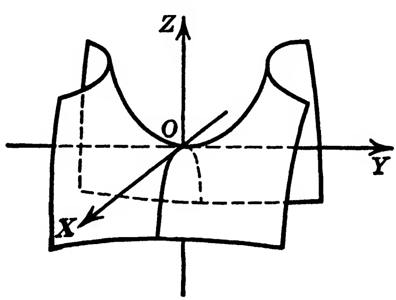

FIGURE 234.

extending upward. Near the origin the surface thus rises on two sides and falls on the other two, and hence has the general shape of a saddle.

If $a = b$ the equation can be changed to the form

$$z = kxy$$

by rotating the axes 45° about OZ.

The cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

The characteristic of this equation is that all terms are of the same degree. If x_1, y_1, z_1 satisfy the equation, kx_1, ky_1, kz_1 also satisfy it. The surface is thus generated by straight lines through the origin.

Equations containing first powers of the coördinates may frequently be reduced to the above forms by completing the squares.

Example. $x^2 - 4y^2 - 2z^2 - 2x + 16y - 4z = 21.$ Completing the squares and dividing by 4, this equation becomes

$$\frac{(x-1)^2}{4} - \frac{(y-2)^2}{1} - \frac{(z+1)^2}{2} = 1.$$

Comparison with equation (3) shows that this equation represents a hyperboloid of two sheets with center $(1, 2, -1)$.

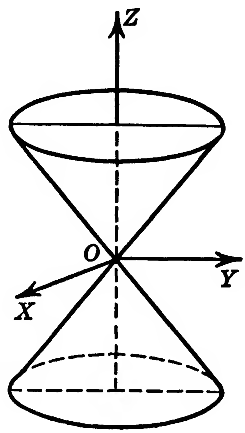

FIGURE 235.

179. Length of Arc. If $f_1(t)$, $f_2(t)$, $f_3(t)$ are given functions and t is a variable parameter, the locus of points determined by the equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t) \tag{1}$$

is a curve. To construct the curve we assign values to t, calculate the corresponding values of x, y, z, and locate the resulting points.

As t increases, the point (x, y, z) may pass more than once through the same position. By restricting the range of t it is usually possible, however, to obtain an arc on which only one value of t corresponds to each point, and conversely. In general we assume that this has been done. Along such an arc there is assigned a positive direction, namely the direction in which points are described as t increases.

The length of such an arc is defined as in §93. We divide the arc into parts by intermediate points and join consecutive points by chords. If the sum of these chords tends to a limit as the number of divisions increases and the chords all tend to zero, the arc is called rectifiable and the limit is its length.

When the functions $f_1(t), f_2(t), f_3(t)$ have continuous derivatives for the values of t under consideration, we readily prove that the

arc is rectifiable. For, if $P(x, y, z)$, $Q(x + \Delta x, y + \Delta y, z + \Delta z)$ are consecutive points of division and t, $t + \Delta t$ the corresponding values of t, the length of the chord between these points is

$$PQ = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

By the mean value theorem (§34)

$$\Delta x = f_1'(\xi_1)\,\Delta t, \quad \Delta y = f_2'(\xi_2)\,\Delta t, \quad \Delta z = f_3'(\xi_3)\,\Delta t,$$

where ξ_1, ξ_2, ξ_3 are values between t and $t + \Delta t$. Thus

$$\Sigma PQ = \Sigma\sqrt{[f_1'(\xi_1)]^2 + [f_2'(\xi_2)]^2 + [f_3'(\xi)_3]^2}\,\Delta t$$

is the sum of the chords, and, as Δt tends to zero, this tends to the limit

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt, \tag{2}$$

a, b being the values of t at the ends of the arc.

At a point where $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}$ are continuous and not all zero, we prove, as in §94, that

$$\lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1. \tag{3}$$

Since $dx = \dfrac{dx}{dt}\,dt$, etc., equation (2) is usually written

$$s = \int_A^B \sqrt{dx^2 + dy^2 + dz^2}. \tag{4}$$

To evaluate this we express x, y, z in terms of a single variable and use as limits the values of this variable at the ends A, B of the arc. Considering the integral as a function of its upper limit, we have

$$ds = \sqrt{dx^2 + dy^2 + dz^2}. \tag{5}$$

Example. Find the length of the curve

$$x = t, \quad y = \frac{1}{t}, \quad z = \sqrt{2}\ln t$$

between the points for which $t = 1$ and $t = 2$.

In this case

$$dx = dt, \quad dy = -\frac{1}{t^2}\,dt, \quad dz = \frac{\sqrt{2}}{t}\,dt,$$

and

$$s = \int_1^2 \sqrt{1 + \frac{1}{t^4} + \frac{2}{t^2}}\,dt = \int_1^2 \left(1 + \frac{1}{t^2}\right) dt = \frac{3}{2}.$$

180. Derivative of a Vector. The derivative of a vector function with respect to a scalar variable is defined in space as in the plane (§110).

Thus, if

$$\mathbf{r} = \overrightarrow{OP}$$

is the vector from a fixed point O to a moving point P,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{1}$$

is the velocity and

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} \tag{2}$$

is the acceleration of P.

Derivatives of sums and products of vectors are obtained by formulas like those for scalars, the only essential difference being that *the order of factors must be preserved in a product if order has significance.*

Suppose, for example, that $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are functions of a scalar variable t and that

$$\mathbf{W} = \mathbf{U} \times \mathbf{V}.$$

When $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ take increments $\Delta\mathbf{U}$, $\Delta\mathbf{V}$, $\Delta\mathbf{W}$ this becomes

$$\begin{aligned} \mathbf{W} + \Delta\mathbf{W} &= (\mathbf{U} + \Delta\mathbf{U}) \times (\mathbf{V} + \Delta\mathbf{V}) \\ &= \mathbf{U} \times \mathbf{V} + \mathbf{U} \times \Delta\mathbf{V} + \Delta\mathbf{U} \times (\mathbf{V} + \Delta\mathbf{V}). \end{aligned}$$

By subtraction we find

$$\Delta\mathbf{W} = \mathbf{U} \times \Delta\mathbf{V} + \Delta\mathbf{U} \times (\mathbf{V} + \Delta\mathbf{V}),$$

whence

$$\frac{\Delta\mathbf{W}}{\Delta t} = \mathbf{U} \times \frac{\Delta\mathbf{V}}{\Delta t} + \frac{\Delta\mathbf{U}}{\Delta t} \times (\mathbf{V} + \Delta\mathbf{V}).$$

If $\mathbf{U}$ and $\mathbf{V}$ are differentiable, $\Delta\mathbf{U}$ and $\Delta\mathbf{V}$ tend to zero with Δt and we have as limit

$$\frac{d}{dt}(\mathbf{U} \times \mathbf{V}) = \mathbf{U} \times \frac{d\mathbf{V}}{dt} + \frac{d\mathbf{U}}{dt} \times \mathbf{V}. \tag{3}$$

In both terms on the right $\mathbf{U}$ and $\mathbf{V}$ have the same positions as on the left. It would not be correct to replace the second term by

$$\mathbf{V} \times \frac{d\mathbf{U}}{dt}.$$

It should be noted that, *if a vector* $\mathbf{V}$ *is of constant length, its derivative is zero or else perpendicular to* $\mathbf{V}$. This may be regarded as geometrically obvious. To prove it analytically we differentiate

$$\mathbf{V}\cdot\mathbf{V} = |\,\mathbf{V}\,|^2 = \text{constant},$$

thus obtaining

$$\mathbf{V}\cdot\frac{d\mathbf{V}}{dt} + \frac{d\mathbf{V}}{dt}\cdot\mathbf{V} = 2\mathbf{V}\cdot\frac{d\mathbf{V}}{dt} = 0.$$

If $|\,\mathbf{V}\,|$ is constant and $\dfrac{d\mathbf{V}}{dt}$ is not zero, this requires that $\mathbf{V}$ and $\dfrac{d\mathbf{V}}{dt}$ be perpendicular.

In particular, *the derivative of a unit vector* $\mathbf{u}$ *is zero or perpendicular to* $\mathbf{u}$.

181. Tangent to a Curve. Principal Normal. Let

$$\mathbf{r} = \overrightarrow{OP} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{1}$$

be the vector from the origin to a point $P(x, y, z)$ on a curve, and s the distance along the curve from a fixed point P_0 to P. As in §111 we show that

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} \tag{2}$$

is a unit vector tangent at P and drawn in the direction in which s increases.

The equation

$$\frac{d\mathbf{r}}{ds} = \mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{ds} + \mathbf{k}\frac{dz}{ds} \tag{3}$$

shows that

$$\frac{dx}{ds} = \cos\alpha, \quad \frac{dy}{ds} = \cos\beta, \quad \frac{dz}{ds} = \cos\gamma \tag{4}$$

are the direction cosines of the tangent at P.

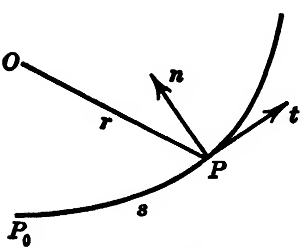

FIGURE 236.

Along the curve, $\mathbf{t}$ is a function of s. Since $\mathbf{t}$ is of unit length its derivative $\dfrac{d\mathbf{t}}{ds}$ is perpendicular to $\mathbf{t}$. This derivative extends along a line called the *principal normal* at P. We usually write

$$\frac{d\mathbf{t}}{ds} = \frac{\mathbf{n}}{\rho}, \tag{5}$$

where $\mathbf{n}$ is a unit vector along the principal normal and ρ is a positive scalar called the *radius of curvature.* The reciprocal of the radius $\frac{1}{\rho}$ is called the *curvature* at P.

If a particle moves along the curve and has the position P at time t, its velocity and acceleration are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{t}\frac{ds}{dt}, \tag{6}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{t}\frac{d^2s}{dt^2} + \frac{\mathbf{n}}{\rho}\left(\frac{ds}{dt}\right)^2. \tag{7}$$

The vector equations for velocity and acceleration in space thus have the same form as in the plane (§112), the principal normal replacing the normal in the plane.

Equation (7) shows that $\frac{d^2\mathbf{r}}{dt^2}$ is the hypotenuse of a triangle with sides $\frac{d^2s}{dt^2}$ and $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$. Thus

$$\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \sqrt{\left(\frac{d^2\mathbf{r}}{dt^2}\right)^2 - \left(\frac{d^2s}{dt^2}\right)^2}$$

and consequently

$$\rho = \frac{\left(\frac{ds}{dt}\right)^2}{\sqrt{\left(\frac{d^2\mathbf{r}}{dt^2}\right)^2 - \left(\frac{d^2s}{dt^2}\right)^2}}. \tag{8}$$

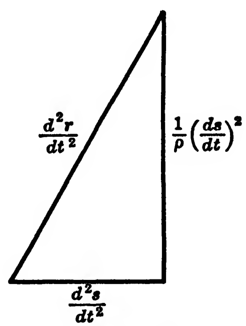

FIGURE 237.

It should be noted that if $\mathbf{r}$ is expressed in terms of any parameter t the derivatives are determined by equations (6) and (7). In equation (8) the variable t need not therefore be the time but may be any parameter in terms of which $\mathbf{r}$ and s are expressed.

Example 1. Find the equations of the tangent line to the helix

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = m\theta$$

at the point where $\theta = 0$.

The point where $\theta = 0$ is $(a, 0, 0)$. The tangent vector at that point has components proportional to

$$\frac{dx}{d\theta} = -a\sin\theta = 0, \quad \frac{dy}{d\theta} = a\cos\theta = a, \quad \frac{dz}{d\theta} = m.$$

The equations of the tangent line are therefore

$$\frac{x-a}{0} = \frac{y}{a} = \frac{z}{m}.$$

These are equivalent to

$$x = a, \quad my = az.$$

Example 2. The coördinates of a moving point at time t are

$$x = t, \quad y = \tfrac{1}{2}t^2, \quad z = \tfrac{1}{3}t^3.$$

Find the tangential and normal components of its acceleration when $t = 1$

The velocity and acceleration are

$$\mathbf{v} = \mathbf{i} + \mathbf{j}t + \mathbf{k}t^2,$$

$$\mathbf{a} = \mathbf{j} + 2\mathbf{k}t,$$

and its speed is

$$\frac{ds}{dt} = \sqrt{1 + t^2 + t^4} = \sqrt{3}.$$

The tangential component of acceleration is

$$a_t = \frac{d^2s}{dt^2} = \frac{t + 2t^3}{\sqrt{1 + t^2 + t^4}} = \sqrt{3},$$

and its normal component is

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{5 - 3} = \sqrt{2}.$$

Example 3. Find the radius of curvature of the helix

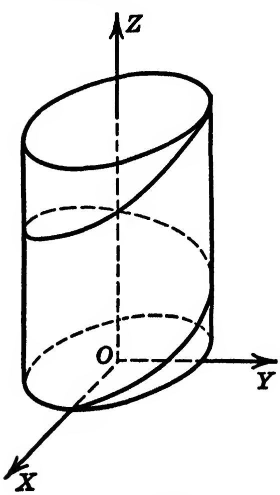

FIGURE 238.

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = m\theta.$$

Taking derivatives with respect to θ, we have

$$\frac{d\mathbf{r}}{d\theta} = -\mathbf{i}a\sin\theta + \mathbf{j}a\cos\theta + \mathbf{k}m$$

$$\frac{d^2\mathbf{r}}{d\theta^2} = -\mathbf{i}a\cos\theta - \mathbf{j}a\sin\theta$$

$$\frac{ds}{d\theta} = \left|\frac{d\mathbf{r}}{d\theta}\right| = \sqrt{a^2 + m^2},$$

$$\frac{d^2s}{d\theta^2} = 0.$$

Substituting in (8) with t replaced by θ, we get

$$\rho = \frac{(a^2 + m^2)}{a}.$$

182. Derivatives in Cylindrical Coördinates. At a point $P(r, \theta, z)$ three coördinate directions are determined, namely the directions in which P moves when two of the coördinates are kept constant and the third increases. The unit vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$ in the directions of increasing r, θ, respectively, evidently have the same values as in the plane (§113). They are therefore functions of θ with derivatives

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta, \quad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r. \tag{1}$$

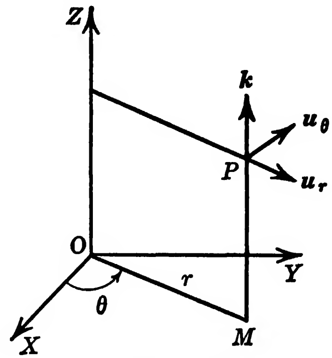

FIGURE 239.

The unit vector in the direction of increasing z is the constant $\mathbf{k}$. By the components of a vector in cylindrical coördinates we mean its components along these unit vectors.

The vector from the origin to P is

$$\mathbf{r} = \overrightarrow{OM} + \overrightarrow{MP} = r\mathbf{u}_r + \mathbf{k}z, \tag{2}$$

where r is the cylindrical coördinate. Along a curve $\mathbf{r}$ is a function of the arc length s from a fixed point of the curve to P. By differentiating (2) and using equations (1) we obtain

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \mathbf{u}_r \frac{dr}{ds} + \mathbf{u}_\theta r \frac{d\theta}{ds} + \mathbf{k} \frac{dz}{ds} \tag{3}$$

as the unit tangent vector. This shows that

$$\frac{dr}{ds}, \quad r\frac{d\theta}{ds}, \quad \frac{dz}{ds}$$

are the cosines of the angles the tangent makes with the unit vectors at P.

By squaring both sides of (3) we have

$$1 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2,$$

whence

$$ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 \tag{4}$$

is the square of the differential of arc.

By differentiating with respect to the time we find

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{u}_r \frac{dr}{dt} + \mathbf{u}_\theta r \frac{d\theta}{dt} + \mathbf{k}\frac{dz}{dt}, \tag{5}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{u}_r\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] + \mathbf{u}_\theta\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) + \mathbf{k}\frac{d^2z}{dt^2}, \tag{6}$$

as the velocity and acceleration of a moving particle. These are merely the polar expressions in the xy-plane plus the components parallel to the z-axis.

Example 1. Find the length of the helix $r = a$, $z = m\theta$, from $\theta = 0$ to $\theta = 2\pi$.

On the helix

$$ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 = (a^2 + m^2)\,d\theta^2.$$

Thus

$$s = \int_0^{2\pi} \sqrt{a^2 + m^2}\,d\theta = 2\pi\sqrt{a^2 + m^2}$$

is the length required.

Example 2. The conical helix

$$r = t, \quad \theta = t, \quad z = t$$

is described on the cone $r = z$. Find the angle at which it intersects the generators of the cone.

Along the helix the unit tangent vector has cylindrical components proportional to

$$\frac{dr}{dt} = 1, \quad r\frac{d\theta}{dt} = t, \quad \frac{dz}{dt} = 1.$$

Assuming the unit tangent drawn in the direction of increasing t, it is then

$$\mathbf{t} = \frac{\mathbf{u}_r + t\mathbf{u}_\theta + \mathbf{k}}{\sqrt{2 + t^2}},$$

the denominator being determined by the fact that $|\mathbf{t}| = 1$. Along a generator of the cone $r = z$, $\theta =$ constant. The unit vector along the generator thus has components proportional to

$$\frac{dr}{dz} = 1, \quad r\frac{d\theta}{dz} = 0, \quad \frac{dz}{dz} = 1.$$

If drawn in the direction of increasing z, it is then

$$\mathbf{u} = \frac{\mathbf{u}_r + \mathbf{k}}{\sqrt{2}}.$$

The angle ϕ between the tangent and the generator is determined by the equation

$$\cos\phi = \mathbf{t}\cdot\mathbf{u} = \frac{1+1}{\sqrt{2}\sqrt{2+t^2}} = \frac{\sqrt{2}}{\sqrt{2+t^2}}.$$

183. Angular Velocity. If a rigid body rotates with angular speed ω about a fixed axis, its *angular velocity* is defined as the vector $\boldsymbol{\omega}$ with magnitude ω which extends along the axis in the direction a right-handed screw advances when subject to the given rotation.

Let O be a fixed point on the axis and $\mathbf{r}$ the vector from O to the point P in the body. This point moves in a circle of radius $|\mathbf{r}|\sin\theta$ with speed $\omega|\mathbf{r}|\sin\theta$, where θ is the angle between $\mathbf{r}$ and $\boldsymbol{\omega}$. Its velocity is therefore

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega}\times\mathbf{r}. \tag{1}$$

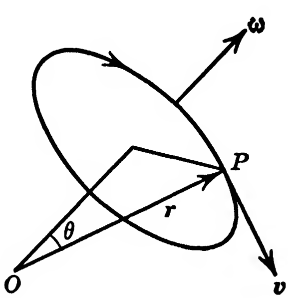

FIGURE 240.

For this vector has the magnitude

$$|\mathbf{v}| = \omega|\mathbf{r}|\sin\theta$$

of the velocity, and the definition of vector product shows that it has the correct direction.

If $\mathbf{r}_1$, $\mathbf{r}_2$ are the vectors from O to two points P_1, P_2 in the body, equation (1) gives

$$\frac{d\mathbf{r}_1}{dt} = \boldsymbol{\omega}\times\mathbf{r}_1, \quad \frac{d\mathbf{r}_2}{dt} = \boldsymbol{\omega}\times\mathbf{r}_2$$

as the velocities of those points. By subtraction we get

$$\frac{d}{dt}(\mathbf{r}_2 - \mathbf{r}_1) = \boldsymbol{\omega}\times(\mathbf{r}_2 - \mathbf{r}_1);$$

that is,

$$\frac{d}{dt}\overrightarrow{P_1P_2} = \boldsymbol{\omega}\times\overrightarrow{P_1P_2}. \tag{2}$$

This equation is satisfied by any vector $\overrightarrow{P_1P_2}$ forming part of the body.

In particular, if the rectangular axes O-XYZ are attached to the rigid body and rotate with it, the unit vectors along those

axes are variable, their derivatives being given by the equations

$$\left.\begin{aligned} \frac{d\mathbf{i}}{dt} &= \boldsymbol{\omega} \times \mathbf{i}, \\ \frac{d\mathbf{j}}{dt} &= \boldsymbol{\omega} \times \mathbf{j}, \\ \frac{d\mathbf{k}}{dt} &= \boldsymbol{\omega} \times \mathbf{k}. \end{aligned}\right\} \tag{3}$$

Example. At a point of latitude θ on the surface of the earth, rectangular axes are used with OZ vertical, OX directed toward the east, and OY toward the north. Find the rates of change of **i**, **j**, **k** due to the rotation of the earth with angular speed ω about its axis.

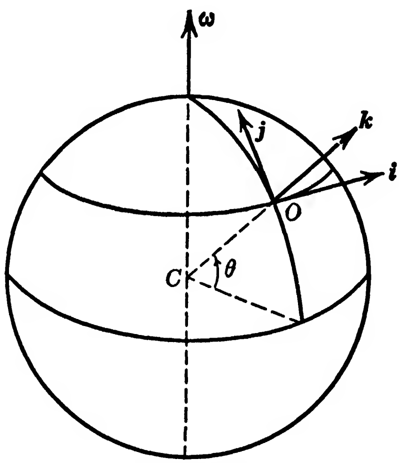

FIGURE 241.

The angular velocity is a vector of magnitude ω directed toward the north along the axis of the earth. This vector is in the yz-plane and makes with the z-axis the angle $90° - \theta$. Thus

$$\boldsymbol{\omega} = \omega(\mathbf{k} \sin \theta + \mathbf{j} \cos \theta).$$

The required rates are therefore

$$\left.\begin{aligned} \frac{d\mathbf{i}}{dt} &= \boldsymbol{\omega} \times \mathbf{i} = \omega(\mathbf{j} \sin \theta - \mathbf{k} \cos \theta), \\ \frac{d\mathbf{j}}{dt} &= \boldsymbol{\omega} \times \mathbf{j} = -\omega \mathbf{i} \sin \theta, \\ \frac{d\mathbf{k}}{dt} &= \boldsymbol{\omega} \times \mathbf{k} = \omega \mathbf{i} \cos \theta. \end{aligned}\right\} \tag{4}$$

PROBLEMS

1. Determine the distance between the point (x, y, z) and the x-axis.

2. Through the point $P(x, y, z)$ a line is drawn perpendicular to the y-axis. Find the coördinates of the point in which it intersects the y-axis.

3. Find the distance between the points $P_1(1, -2, 3)$ and $P_2(-1, -3, 4)$.

4. Find the unit vector having the same direction as

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

5. The vectors

$$\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$\mathbf{B} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

extend from the origin. Show that the line joining their ends is parallel to the xy-plane, and find its length.

6. By showing that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel prove that the points $A(1, 2, -1)$, $B(0, 5, 1)$, $C(-2, 11, 5)$ are on a line.

7. If $\mathbf{A}$, $\mathbf{B}$ are vectors and x, y, scalars, show that $\mathbf{A}$, $\mathbf{B}$, and $x\mathbf{A} + y\mathbf{B}$ are parallel to a plane.

8. AE is the diagonal of a parallelepiped with edges AB, AC, AD. Given $A(1, 1, 0)$, $B(2, 3, 0)$, $C(3, 0, 1)$, $D(2, 1, 4)$, find the coördinates of E.

9. Given $A(3, 4, 5)$, $B(1, 2, -1)$, $C(3, 0, 2)$, $D(5, 2, 6)$, find the middle points of AB and CD. Then find the middle point P of the line joining the two middle points. Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0.$$

10. The vectors from the origin to the points A, B are

$$\mathbf{A} = 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}, \quad \mathbf{B} = 2\mathbf{i} + \mathbf{j} - 6\mathbf{k}.$$

Find the vector from the origin to the point on the line AB, one-third of the way from A to B.

11. If three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ satisfy an equation

$$m\mathbf{A} + n\mathbf{B} + p\mathbf{C} = 0$$

with coefficients m, n, p not all zero, show that the plane parallel to two of the vectors is parallel to the third, and consequently that the three vectors are parallel to a plane.

12. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{P}$ are vectors from the origin to the points A, B, C, P and

$$\mathbf{P} = m\mathbf{A} + n\mathbf{B} + p\mathbf{C},$$

$$1 = m + n + p,$$

show that the vectors $\overrightarrow{AP}$, $\overrightarrow{BP}$, $\overrightarrow{CP}$ are in a plane, and consequently that the four points A, B, C, P are in a plane.

13. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are vectors from the origin to the points A, B, C, D, and

$$\overrightarrow{AB} = m\overrightarrow{CD},$$

show that

$$\frac{\mathbf{B} - m\mathbf{D}}{1 - m} = \frac{\mathbf{A} - m\mathbf{C}}{1 - m},$$

and consequently that this is the vector from the origin to the intersection of AC and BD.

14. The points A, B, C are vertices of an equilateral triangle of side x. Determine the products, (*a*) $\overrightarrow{AB}\cdot\overrightarrow{BC}$, (*b*) $\overrightarrow{AB}\cdot\overrightarrow{AC}$.

15. Show that the vectors

$$\mathbf{A} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k},$$

$$\mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

are perpendicular.

16. Show that the vectors

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},$$

$$\mathbf{B} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k},$$

$$\mathbf{C} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$$

form the sides of a right triangle.

17. If $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are mutually perpendicular unit vectors, show that

$$(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3)^2 = 3.$$

18. Show that

$$(\mathbf{B}\cdot\mathbf{B})\mathbf{A} - (\mathbf{A}\cdot\mathbf{B})\mathbf{B}$$

is perpendicular to $\mathbf{B}$.

19. Given

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$

$$\mathbf{B} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k},$$

find a vector in the plane of $\mathbf{A}$ and $\mathbf{B}$ perpendicular to the x-axis.

20. Given

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},$$

$$\mathbf{B} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

$$\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

find a vector parallel to the plane of $\mathbf{B}$ and $\mathbf{C}$, and perpendicular to $\mathbf{A}$.

21. Given

$$\mathbf{A} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k},$$

$$\mathbf{B} = 3\mathbf{i} - 4\mathbf{k},$$

find the component of $\mathbf{A}$ along $\mathbf{B}$.

22. Find the component of the vector

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

along the vector

$$\mathbf{B} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k},$$

23. By squaring both sides of the equation

$$\mathbf{a} = \mathbf{b} - \mathbf{c}$$

and interpreting the result geometrically, prove the formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

expressing the square of one side of a triangle in terms of the other two sides and the included angle.

24. Determine the angles the vector

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$$

makes with the coördinate axes.

25. A line through the origin makes angles of 60° with both OX and OY. Determine the angle between this line and the z-axis.

26. Find the angle between the vectors

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + 2\mathbf{k},$$

$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

27. Find the angle between the vectors

$$\mathbf{a} = 4\mathbf{i} + \mathbf{j} + \mathbf{k},$$

$$\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}.$$

28. Determine the interior angles of the triangle with vertices $A(0, 2, -2)$, $B(4, 1, -1)$, $C(5, 0, 3)$.

29. Determine the interior angles of the triangle with vertices $A(1, -2, 1)$, $B(2, 0, -1)$, $C(2, -3, 5)$.

30. Show that

$$\mathbf{i}' = \tfrac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}),$$

$$\mathbf{j}' = \tfrac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

$$\mathbf{k}' = \tfrac{1}{3}(2\mathbf{i} - 2\mathbf{j} - \mathbf{k})$$

form a set of mutually perpendicular unit vectors. If new axes OX', OY', OZ' have the directions of these unit vectors, find the equations expressing x, y, z in terms of x', y', z'.

31. Given

$$\mathbf{A} = \mathbf{i} - \mathbf{j} + 2\mathbf{k},$$

$$\mathbf{B} = 2\mathbf{i} + \mathbf{j} - \mathbf{k},$$

find a vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

32. Find the area of the parallelogram two edges of which are formed by the vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = 2\mathbf{j} - 4\mathbf{k}$.

33. Find the area of the triangle formed by the points $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$.

34. Find the area of the triangle with vertices $A(1, 2, 0)$, $B(3, 3, 0)$, $C(5, -1, 0)$.

35. In problem 30 show that $\mathbf{i}' \times \mathbf{j}' = \mathbf{k}'$ and that the vector

$$\mathbf{i} \times \mathbf{i}' + \mathbf{j} \times \mathbf{j}' + \mathbf{k} \times \mathbf{k}'$$

makes equal angles with $\mathbf{i}$ and $\mathbf{i}'$, $\mathbf{j}$ and $\mathbf{j}'$, and $\mathbf{k}$ and $\mathbf{k}'$.

36. Show that $\mathbf{A}$ is zero if $\mathbf{A} \cdot \mathbf{B} = 0$ or $\mathbf{A} \times \mathbf{B} = 0$ for all values of $\mathbf{B}$.

37. Show that $\mathbf{A}$ is zero if $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \times \mathbf{B} = 0$ and $\mathbf{B}$ is not zero.

38. The vectors from the origin to the points A, B, C are

$$\mathbf{A} = \mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$\mathbf{B} = 3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},$$

$$\mathbf{C} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}.$$

Find a vector $\mathbf{N}$ perpendicular to the plane ABC. By projecting $\mathbf{A}$ upon $\mathbf{N}$ find the distance from the origin to the plane.

39. Given $A(1, 0, 0)$, $B(0, 1, 1)$, $C(1, -1, 1)$, find the vector which extends from the origin to a point on the plane ABC and is perpendicular to that plane.

40. Given

$$\mathbf{A} = \mathbf{i} - \mathbf{j} + 2\mathbf{k},$$

$$\mathbf{B} = \mathbf{i} - \mathbf{j},$$

$$\mathbf{C} = \mathbf{j} - \mathbf{k},$$

find the vector which is the projection of $\mathbf{A}$ upon a plane parallel to $\mathbf{B}$ and $\mathbf{C}$.

41. Given $A(1, 2, 3)$, $B(0, 1, 0)$, $C(1, 1, 0)$, find the projection of the point C on the plane OAB.

42. Find the distance between the line through $A(1, 1, 0)$, $B(1, 0, 1)$ and that through $C(0, 1, 1)$, $D(1, 0, 0)$.

43. Find the distance between the line through $A(1, -2, -5)$, $B(-1, 1, 1)$ and that through $C(4, 5, 1)$, $D(1, -1, 3)$.

44. The force

$$\mathbf{F} = \mathbf{i}a + \mathbf{j}b + \mathbf{k}c$$

is applied at $P(x, y, z)$. Find its moment about the origin and its moment about the x-axis.

45. The force

$$\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

is applied at the point $P(3, -2, 0)$. Find its moment about the point $(2, 3, 1)$.

46. The force

$$\mathbf{F} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

is applied at the origin. Find its moment about the axis through $(1, 2, 3)$ parallel to OX.

47. A force

$$\mathbf{F} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

is applied at the point $(1, 2, 3)$. Find its moment about an axis through the origin directed toward the point $(2, 3, 1)$.

48. Show that a force is zero if it has equal moments about two points not on a line parallel to the force.

49. Show that the identity

$$\mathbf{V} = \tfrac{1}{2}[\mathbf{i} \times (\mathbf{V} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{V} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{V} \times \mathbf{k})]$$

is satisfied by every vector $\mathbf{V}$.

50. Find the volume of the parallelepiped three of whose edges are $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, and $\mathbf{k} + \mathbf{i}$.

51. Given $A(-1, 1, 2)$, $B(0, 2, 3)$, $C(1, 1, 1)$, $D(-1, 3, 3)$, find the volume of the parallelepiped three of whose edges are AB, AC, AD.

52. Find the volume of the tetrahedron with vertices $A(1, 1, 0)$, $B(3, 2, -1)$, $C(-2, 1, 1)$, and $D(2, -1, 0)$.

53. By means of products express the condition that three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be parallel to a plane.

54. By means of products express the condition that the plane containing the vectors $\mathbf{A}$ and $\mathbf{B}$ be perpendicular to that containing $\mathbf{C}$ and $\mathbf{D}$.

55. By means of products find a vector which is in the plane of $\mathbf{B}$ and $\mathbf{C}$ and is perpendicular to $\mathbf{A}$.

56. By means of products express the condition that four vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ be parallel to a plane, assuming that no two of the vectors are parallel.

57. If $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ is not zero, show that

$$\mathbf{V} = \frac{l\mathbf{B} \times \mathbf{C} + m\mathbf{C} \times \mathbf{A} + n\mathbf{A} \times \mathbf{B}}{\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}}$$

is the solution of the equations

$$\mathbf{V} \cdot \mathbf{A} = l, \quad \mathbf{V} \cdot \mathbf{B} = m, \quad \mathbf{V} \cdot \mathbf{C} = n.$$

58. If $\mathbf{A} \cdot \mathbf{B} = 0$ and $\mathbf{A}$ is not zero, show that

$$\mathbf{V} = \frac{\mathbf{A} \times \mathbf{B} + p\mathbf{A}}{\mathbf{A} \cdot \mathbf{A}}$$

is the solution of the equations

$$\mathbf{V} \times \mathbf{A} = \mathbf{B}, \quad \mathbf{V} \cdot \mathbf{A} = p.$$

59. Assuming that $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ is not zero, find the vector $\mathbf{V}$ which has the components l, m, n along $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

60. Given

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

$$\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

$$\mathbf{C} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

find the vector $\mathbf{V}$ with components 1 along $\mathbf{A}$, 2 along $\mathbf{B}$, and 3 along $\mathbf{C}$.

61. If $\mathbf{A} = \overrightarrow{PP'}$, $\mathbf{B} = \overrightarrow{PQ}$, $\mathbf{C} = \overrightarrow{P'Q'}$, and $\mathbf{B} \times \mathbf{C}$ is not zero, show that

$$h = \frac{\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}}{|\mathbf{B} \times \mathbf{C}|}$$

is the perpendicular distance between the lines PQ and $P'Q'$.

62. Prove the formula

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B} \times \mathbf{D})\mathbf{C} - (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})\mathbf{D}.$$

Determine the loci represented by the following equations:

63. $2x + 3y = 6.$

64. $z^2 = 4x.$

65. $y^2 + z^2 = 4.$

66. $x^2 - y^2 = 1.$

67. Find the equation of the sphere with center (2, 0, 1) and radius 3.

68. Find the equation of the circular cylinder with axis OY and radius r.

69. Find the equation of the plane which is parallel to the x-axis and passes through the points (0, 1, 2), (0, 3, 4).

70. Find the projection of the curve $x^2 + y^2 = a^2$, $y^2 + z^2 = a^2$ on the xz-plane.

71. Show that the curve

$$x^2 + y^2 = 2z, \quad z = x + 4$$

is on the cylinder

$$(x - 1)^2 + y^2 = 9.$$

72. Find the equation of the cylinder which passes through the curve

$$x = 2t, \quad y = t^2, \quad z = t^3$$

and has generators parallel to OY.

73. Find the equation of the plane which passes through (−1, 2, 1) and is perpendicular to the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

74. Find the equation of the plane through the origin if $\alpha = 60°$, $\beta = 45°$, $\gamma = 60°$ are the direction angles of its normal.

75. Show that

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

represents the plane with intercepts a, b, c on OX, OY, OZ, respectively.

76. Find the equation of the plane which passes through (2, 1, −3) and is parallel to the plane $3x - y + 2z = 4$.

77. Find the equation of the plane through the points $A(0, 1, 2)$, $B(-1, 0, 1)$, $C(2, 3, 0)$.

78. Show that the planes $x - 2y + z = 3$, $3x - 6y + 3z = 1$ are parallel.

79. Show that the planes $x + 2y - 2z = 5$, $2x + y + 2z = 7$ are perpendicular.

80. Find the angle between the planes $x + y = 1$, $y + z = 2$.

81. Find the angle between the planes $x + 2y + 2z = 0$, $x - 4y + 8z = 0$.

82. Find the angle between the vector $7\mathbf{j} + 8\mathbf{k} - 7\mathbf{i}$ and the plane $x + 8y + 4z = 9$.

83. Find the equations of the line which passes through the points $P_1(2, -1, 3)$, $P_2(4, 2, -1)$.

84. Find the equations of the line which passes through the points (2, 3, −1), and (3, 4, 2).

85. Find the equations of the line which passes through (3, 1, −2) and is parallel to the vector $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

86. Find the equations of the line which passes through $(2, 1, -3)$ and is perpendicular to the plane $4x - 3y + z = 5$.

87. Find the equation of the plane which is parallel to the y-axis and passes through the line $x + 2y + 3z = 4$, $2x + y + z = 2$.

88. Find the equation of the plane which passes through the line $x + y = 3$, $2y + 3z = 4$ and is parallel to the vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

89. Find the angle between the line

$$\frac{x-1}{4} = \frac{y+1}{-1} = \frac{z}{1}$$

and the line

$$\frac{x}{5} = \frac{y-1}{-2} = \frac{z+1}{5}.$$

90. Find the angle between the lines $x + y - z = 0$, $x + z = 0$ and $x - y = 1$, $x - 3y + z = 0$.

91. Find the angle between the lines $2x - 2y - z = 0$, $2x + y + 2z = 3$ and $3x - y - z = 0$, $2x - 2y - z = 2$.

Determine the loci represented in cylindrical coördinates by the following equations:

92. $r = a \cos \theta$.

93. $r = a \sec \theta$.

94. $\theta = \dfrac{\pi}{4}$.

95. $r = az + b$.

96. Find the equation of the paraboloid generated by rotating the parabola $y^2 = 4px$ about the x-axis.

97. Find the equation of the surface generated by rotating the hyperbola $xy = 1$ about the y-axis.

98. Find the equation of the spheroid generated by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

99. The line $2x + 3y = 6$ is rotated about the y-axis. Find the equation of the cone generated.

100. Find the equation of the cone with vertex $(1, 2, 3)$ and axis parallel to OZ, if the generators make angles of $45°$ with the axis.

101. In cylindrical coördinates determine the equation of the surface generated by rotating the circle $(z - 1)^2 + (x - 2)^2 = 16$ about the z-axis.

Determine the quadric surfaces represented by the following equations:

102. $x^2 + 2y^2 + 3z^2 = 6$.

103. $(x - 1)^2 + 2(y + 1)^2 + 3(z - 2)^2 = 6$.

104. $x^2 + 4(y^2 + z^2) = 12$.

105. $x^2 - y^2 + z^2 = 1$.

106. $\dfrac{x^2}{4} - \dfrac{y^2}{9} - \dfrac{z^2}{1} = 1$.

107. $x^2 - y^2 + z^2 - 2x + 4y = 4$

108. $x^2 - y^2 = z^2$.

109. $x^2 + z^2 = y$.

110. $z = 2xy$.

111. $z = xy + x + y$.

112. $y^2 = z^2$.

113. $r = a \sin \theta$.

114. $r^2 + 4z^2 = 4$.

115. $2r + 3z = 6$.

116. Find the length of the curve

$$x = \sin\theta, \quad y = \theta, \quad z = 1 - \cos\theta$$

from $\theta = 0$ to $\theta = 2\pi$.

117. Find the length of the curve

$$x = t, \quad y = \frac{t^2}{\sqrt{2}}, \quad z = \tfrac{1}{3}t^3$$

from $t = 0$ to $t = 3$.

118. Find the length of the curve

$$x = t, \quad y = 3t^2, \quad z = 6t^3$$

from $t = 0$ to $t = 2$.

119. Find the length of the curve

$$y = \ln \sec x, \quad z = \ln(\sec x + \tan x)$$

from $x = 0$ to $x = \frac{\pi}{4}$.

120. Find the derivative of $\mathbf{F} \times \frac{d\mathbf{F}}{dt}$ with respect to t.

121. Find the derivative of $\mathbf{F} \cdot \frac{d\mathbf{F}}{dt} \times \frac{d^2\mathbf{F}}{dt^2}$ with respect to t.

122. Find the equations of the tangent line to the curve

$$x = t, \quad y = t^2, \quad z = t^3$$

at the point where $t = 2$.

123. Find the equations of the tangent to the curve

$$x = t - 1, \quad y = t^2 + t, \quad z = t^4$$

at the point where it crosses the yz-plane.

124. The helix

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = m\theta$$

winds around the cylinder $x^2 + y^2 = a^2$. Find the angle at which it intersects the generators of the cylinder.

125. On the curve $y = \sin x$, $z = \cos x$ find the unit vectors $\mathbf{t}$ and $\mathbf{n}$, and the radius of curvature.

126. On the curve

$$y = \ln \sec x, \quad z = \ln(\sec x + \tan x)$$

find the unit vectors $\mathbf{t}$ and $\mathbf{n}$, and the radius of curvature.

127. A particle moving in a curve has the coördinates

$$x = t, \quad y = \tfrac{1}{2}t^2, \quad z = \tfrac{1}{3}t^3$$

at time t. Find its velocity and acceleration, and the radius of curvature of the path at the point where $t = 1$.

128. Find the length of the curve $r = az$, $z = e^\theta$ from $\theta = 0$ to $\theta = 1$.

129. Find the length of the curve

$$r = a\cos t, \quad \theta = \sec t, \quad z = a\sin t$$

from $t = 0$ to $t = \frac{\pi}{6}$.

130. The curve $r = \sin\theta$, $z = \ln\sec\theta$ is on the cylinder $r = \sin\theta$. Find the angle at which it intersects the generators of the cylinder.

131. A point P moves with constant velocity

$$\frac{dr}{dt} = a, \quad \frac{dz}{dt} = b$$

in the rz-plane while that plane rotates about the z-axis with constant angular velocity ω. Find the acceleration of P.

132. A particle moving in a conical helix has the position

$$r = t, \quad \theta = t, \quad z = t$$

at time t. Find its velocity and acceleration and the radius of curvature of the path at the point where $t = 0$.

133. A rigid body including the coördinate axes rotates with angular velocity

$$\boldsymbol{\omega} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

about an axis through the origin. Find the velocity of the point $(1, 2, -1)$ and the rate of change of the unit vectors **i**, **j**, **k**.

134. If $\dfrac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}$, $\dfrac{d\mathbf{B}}{dt} = \boldsymbol{\omega} \times \mathbf{B}$, show that

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \boldsymbol{\omega} \times (\mathbf{A} \times \mathbf{B}).$$

135. The position of a moving point P is determined by coördinates x, y, z with respect to axes which rotate as in Figure 241. The vector from the center of the earth to P is

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z,$$

where $\mathbf{r}_0 = \overrightarrow{CO}$. The acceleration of P is $\dfrac{d^2\mathbf{r}}{dt^2}$, the differentiations being performed with $\mathbf{r}_0$, **i**, **j**, **k**, x, y, z all variable. At the instant when P passes through O, express its acceleration in terms of the acceleration $\dfrac{d^2\mathbf{r}_0}{dt^2}$ of O, and the velocity

$$\mathbf{v} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}$$

and acceleration

$$\mathbf{a} = \mathbf{i}\frac{d^2x}{dt^2} + \mathbf{j}\frac{d^2y}{dt^2} + \mathbf{k}\frac{d^2z}{dt^2}$$

of P with respect to the moving axes.

CHAPTER XVI

PARTIAL DIFFERENTIATION

184. Functions of Two or More Variables. Continuity. A function $f(x, y)$ is called continuous at $x = x_0$, $y = y_0$ if $f(x_0, y_0)$ has a definite value and $f(x, y)$ tends to $f(x_0, y_0)$ as limit whenever x tends to x_0 and y tends to y_0. The analytic condition for this is that to each positive number ϵ there correspond a positive number δ such that

$$|f(x, y) - f(x_0, y_0)| < \epsilon$$

whenever $|x - x_0|$ and $|y - y_0|$ are both less than δ.

Continuity of a vector function $\mathbf{F}(x, y)$ is defined in the same way, vector $\mathbf{F}(x, y)$ merely replacing scalar $f(x, y)$ in the above statement. Similar definitions may be given in space or on a range of any finite number of variables.

185. Partial Derivative. Let $f(x, y)$ be a function of two independent variables x and y. When y is kept constant, $f(x, y)$ is a function of x only. The derivative of this function with respect to x is called the *partial derivative* of $f(x, y)$ with respect to x, and is denoted by

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x(x, y).$$

Similarly, if we differentiate with respect to y with x constant, we get the partial derivative with respect to y, denoted by

$$\frac{\partial f}{\partial y} \quad \text{or} \quad f_y(x, y).$$

For example, if

$$f(x, y) = x^2 + xy + 2y^2,$$

we have

$$\frac{\partial f}{\partial x} = 2x + y, \quad \frac{\partial f}{\partial y} = x + 4y.$$

Likewise, if a function, scalar or vector, depends on any number of independent variables, its partial derivative with respect to one is obtained by differentiating with all the other variables constant.

186. Dependent Variables. It often happens that some of the variables are functions of others. For example, let

$$u = xyz,$$

and let z be a function of x and y. When y is constant, z is a function of x and

$$\frac{\partial u}{\partial x} = xy\frac{\partial z}{\partial x} + yz.$$

Similarly

$$\frac{\partial u}{\partial y} = xy\frac{\partial z}{\partial y} + xz.$$

If, however, we consider z an independent variable,

$$\frac{\partial u}{\partial x} = yz, \quad \frac{\partial u}{\partial y} = xz.$$

The value of a partial derivative thus depends on what is constant as well as on what is variable during the differentiation.

The variables kept constant during differentiation are sometimes indicated by subscripts. Thus, in the above example

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} = yz, \quad \left(\frac{\partial u}{\partial x}\right)_y = xy\frac{\partial z}{\partial x} + yz.$$

Example. If a, b are sides of a right triangle with opposite angles A, B and hypotenuse c, find $\left(\frac{\partial a}{\partial c}\right)_A$ and $\left(\frac{\partial a}{\partial c}\right)_b$.

From the diagram (Figure 242) it is seen that $a = c \sin A$. Thus

$$\left(\frac{\partial a}{\partial c}\right)_A = \sin A.$$

Also $a = \sqrt{c^2 - b^2}$, and so

$$\left(\frac{\partial a}{\partial c}\right)_b = \frac{c}{\sqrt{c^2 - b^2}} = \frac{1}{\sin A}.$$

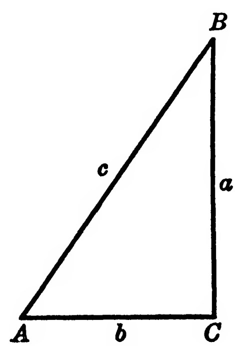

FIGURE 242.

187. Geometrical Representation. If x, y, z are taken as rectangular coördinates of a point $P(x, y, z)$, the equation

$$z = f(x, y)$$

represents a surface. The points with given value of y form a curve AB in which a plane $y =$ constant cuts the surface. On this curve z is the vertical and x the horizontal coördinate, and

$$\frac{\partial z}{\partial x} = f_x(x, y)$$

is the slope with respect to the xy-plane. Similarly, the locus of points with given x is a curve CD with slope

$$\frac{\partial z}{\partial y} = f_y(x, y).$$

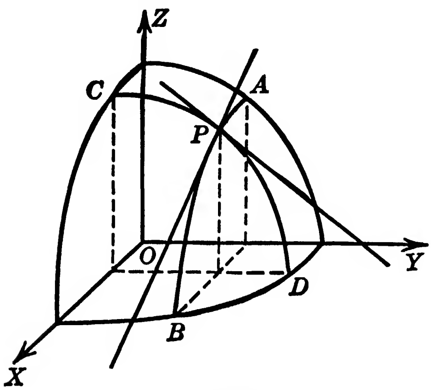

FIGURE 243.

Through any point $P(x, y, z)$ of the surface there pass two curves $y =$ constant, $x =$ constant, and these curves have the slopes $f_x(x, y)$, $f_y(x, y)$ at P.

At a top or bottom point of the surface the tangents to the curves AB, CD are horizontal and

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0.$$

Example. Find the highest point on the surface

$$z = 2x + 4y - x^2 - y^2.$$

At the highest point

$$\frac{\partial z}{\partial x} = 2 - 2x = 0, \quad \frac{\partial z}{\partial y} = 4 - 2y = 0.$$

Consequently $x = 1$, $y = 2$. These values substituted in the equation of the surface give $z = 5$. The point required is therefore (1, 2, 5).

188. Increment. Let $f(x, y)$ be a function of x and y with partial derivatives $f_x(x, y)$ and $f_y(x, y)$. When x, y change to $x + \Delta x$, $y + \Delta y$, the increment of $f(x, y)$ is

$$\Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{1}$$

This can be written

$$\Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y). \quad (2)$$

By the mean value theorem (§34)

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \Delta x\, f_x(x_1, y + \Delta y), \quad (3)$$

where x_1 is between x and $x + \Delta x$. Similarly,

$$f(x, y + \Delta y) - f(x, y) = \Delta y\, f_y(x, y_1), \quad (4)$$

where y_1 is between y and $y + \Delta y$. These values being substituted, (2) becomes

$$\Delta f(x, y) = \Delta x\, f_x(x_1, y + \Delta y) + \Delta y\, f_y(x, y_1). \quad (5)$$

As Δx and Δy approach zero, x_1 approaches x and y_1 approaches y. If the partial derivatives are continuous,

$$f_x(x_1, y + \Delta y) = f_x(x, y) + \epsilon_1,$$

$$f_y(x, y_1) = f_y(x, y) + \epsilon_2,$$

ϵ_1 and ϵ_2 tending to zero with Δx and Δy. In virtue of (5) we have thus proved:

At a point where the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are continuous,

$$\Delta f(x, y) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1\, \Delta x + \epsilon_2\, \Delta y, \quad (6)$$

ϵ_1 and ϵ_2 tending to zero when Δx and Δy tend to zero.

The expression

$$\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \quad (7)$$

is called the *principal part* of Δf. It differs from Δf by the amount $\epsilon_1\, \Delta x + \epsilon_2\, \Delta y$. As Δx and Δy tend to zero, if the partial derivatives are continuous, ϵ_1 and ϵ_2 tend to zero, and so this difference becomes an infinitesimal of higher order than the larger of the increments Δx, Δy. When Δx and Δy are sufficiently small the principal part thus gives a satisfactory approximation for the increment.

Analogous results can be obtained for any number of independent variables. For example, if $f(x, y, z)$ is a function of three independent variables x, y, z, the principal part of Δf is

$$\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z.$$

At a point where the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ are continuous,

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y + \epsilon_3\,\Delta z,$$

$\epsilon_1, \epsilon_2, \epsilon_3$ tending to zero when Δx, Δy, and Δz tend to zero.

189. Total Differential. If x and y are the independent variables, the principal part of $\Delta f(x, y)$, written

$$df(x, y) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y, \tag{1}$$

is called the differential, or *total differential*, of $f(x, y)$.

This definition applies to any function of x and y. The particular values $f = x, f = y$ give

$$dx = \Delta x, \quad dy = \Delta y. \tag{2}$$

Thus (1) can also be written

$$df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{3}$$

The quantities

$$\frac{\partial f}{\partial x}dx, \quad \frac{\partial f}{\partial y}dy$$

are called partial differentials. Equations (2) and (3) state that *the differentials of the independent variables are equal to their increments, and the differential of a function is equal to the sum of the partial differentials obtained by letting the variables change one at a time.*

Similar results are obtained for functions of any number of variables. Thus, if x, y, z are independent variables,

$$df(x, y, z) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z.$$

The particular values $f = x, f = y, f = z$ give

$$dx = \Delta x, \quad dy = \Delta y, \quad dz = \Delta z,$$

and hence

$$df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz. \tag{4}$$

Example 1. Find the total differential of the function

$$u = x^2y + xy^2 - xyz - z^3.$$

By equation (4)

$$\begin{aligned} du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \\ &= (2xy + y^2 - yz)\, dx + (x^2 + 2xy - xz)\, dy - (xy + 3z^2)\, dz. \end{aligned}$$

Example 2. Find the increment and the differential of

$$r = \sqrt{x^2 + y^2},$$

if x and y are the independent variables and $x = 3$, $y = 4$, $\Delta x = 0.1$, $\Delta y = -0.2$.

The differential is

$$dr = \frac{\partial r}{\partial x} \Delta x + \frac{\partial r}{\partial y} \Delta y = \frac{x\, \Delta x + y\, \Delta y}{\sqrt{x^2 + y^2}} = -0.1.$$

The increment is

$$\Delta r = \sqrt{(3.1)^2 + (3.8)^2} - \sqrt{3^2 + 4^2} = -0.096.$$

Example 3. Find the maximum error in

$$u = \frac{xy}{z^2}$$

due to errors of 1% in x, y, and z.

The error in u due to small errors Δx, Δy, Δz in x, y, z is Δu. If the increments are sufficiently small and z not too near zero, this will be approximately

$$du = \frac{y}{z^2} dx + \frac{x}{z^2} dy - \frac{2xy}{z^3} dz.$$

Dividing the left side by u, the right side by $\frac{xy}{z^2}$, we get

$$\frac{du}{u} = \frac{dx}{x} + \frac{dy}{y} - \frac{2dz}{z}.$$

Now $\frac{dx}{x}$ is the relative error in x. The equation just obtained thus states that the relative error in u is the sum of the relative errors in x and y

minus twice that in z. Since the algebraic signs could be such that all the errors add, the maximum error in u due to errors of 1% in x, y, z is then 4%.

190. Derivative of a Composite Function. Let $f(x, y)$ be a function of two variables x, y, and let x, y be functions of two variables s, t. Then $f(x, y)$ is a function of s and t. To determine the partial derivative of this function with respect to t we leave s constant and change t to $t + \Delta t$. The variables x, y change to $x + \Delta x$, $y + \Delta y$, and, assuming the partial derivatives continuous,

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{1}$$

is the resulting increment in $f(x, y)$. Consequently

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

When Δt approaches zero, Δx and Δy, and so ϵ_1 and ϵ_2, tend to zero, and this gives as limit

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \tag{2}$$

If x or y is a function of t only, the partial derivative $\frac{\partial x}{\partial t}$ or $\frac{\partial y}{\partial t}$ becomes a total derivative $\frac{dx}{dt}$ or $\frac{dy}{dt}$. If both x and y are functions of t only, $f(x, y)$ is a function of t with derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{3}$$

Similar results are obtained for any number of variables. Thus if $f(x, y, z)$ is a function of three variables x, y, z and these are functions of t and other variables,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}, \tag{4}$$

provided that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are continuous functions of x, y, z.

The term

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}$$

is obtained by differentiating $f(x, y, z)$ with respect to t, leaving y and z constant. Equations (2) and (4) state that, *if f is a function of several variables, $\frac{\partial f}{\partial t}$ is obtained by differentiating as if only one of these were variable at a time and adding the results.*

Example 1. If $f(x, y) = x^4 - x^2y + y^2$, $x = t$, $y = t^2$, find $\frac{df}{dt}$.

By equation (3)

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (4x^3 - 2xy)\frac{dx}{dt} + (2y - x^2)\frac{dy}{dt} \\ &= 2t^3 + t^2 \cdot 2t \\ &= 4t^3.\end{aligned}$$

This may be checked by substituting $x = t$, $y = t^2$ before differentiating. In this way we obtain

$$f(x, y) = t^4 - t^4 + t^4 = t^4$$

and

$$\frac{df}{dt} = 4t^3$$

as before.

Example 2. If x and y are independent variables, z is a function of x and y, and $u = f(x, y, z)$, find $\frac{\partial u}{\partial x}$.

Since $\frac{\partial y}{\partial x}$ is zero, equation (4) gives

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}.$$

Example 3. If $y = x^x$, find $\frac{dy}{dx}$.

The function x^x may be considered as a function of two variables, the lower x and the upper x. If the upper x is kept constant and the lower allowed to vary, the derivative (as in x^n) is

$$xx^{x-1} = x^x.$$

If the lower x is kept constant while the upper varies, the derivative (as in a^x) is

$$x^x \ln x.$$

When both vary the derivative is the sum

$$\frac{dy}{dx} = x^x + x^x \ln x.$$

Compare §88, example.

Example 4. If the substitution

$$x = 2u + 3v, \quad y = 3u - 2v$$

changes $f(x, y)$ into $F(u, v)$, determine $\dfrac{\partial F}{\partial u}$ in terms of derivatives of $f(x, y)$ with respect to x and y.

By assumption

$$F(u, v) = f(x, y).$$

Because of the equations connecting x, y and u, v, both sides of this equation may be regarded as functions of u and v. Differentiating both sides with v constant, we get

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2\frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y}.$$

Example 5. If x and y are independent variables and $u = f(2x - 3y)$, show that

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$

For convenience of notation we take $2x - 3y = v$ as a new variable. We then have $u = f(v)$,

$$\frac{\partial u}{\partial x} = \frac{df}{dv}\frac{\partial v}{\partial x} = 2\frac{df}{dv},$$

$$\frac{\partial u}{\partial y} = \frac{df}{dv}\frac{\partial v}{\partial y} = -3\frac{df}{dv},$$

whence

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0,$$

which was to be shown.

191. Homogeneous Functions. A function is said to be *homogeneous of the nth degree* in a given set of variables if multiplication of each variable by k multiplies the function by k^n.

Thus $f(x, y)$ is a homogeneous function of the nth degree in x and y if

$$f(kx, ky) = k^n f(x, y) \tag{1}$$

for all values of k. Since k is arbitrary we can differentiate both sides of this equation with respect to k, leaving x and y constant. For convenience in doing this let

$$u = kx, \quad v = ky.$$

The equation then becomes

$$f(u, v) = k^n f(x, y).$$

By equation (2), §190, we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial k} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial k} = nk^{n-1}f(x, y);$$

that is,

$$x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} = nk^{n-1}f(x, y). \tag{2}$$

This holds for all values of k. Putting $k = 1$, we get $u = x, v = y$,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial y}$$

and (2) becomes

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \tag{3}$$

In a similar way we show that, if $f(x_1, x_2, \cdots, x_m)$ *is a homogeneous function of the nth degree in* $x_1, x_2, \cdots, x_m$,

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_m\frac{\partial f}{\partial x_m} = nf. \tag{4}$$

This is known as *Euler's theorem on homogeneous functions.*

Example 1. The function

$$f = \sqrt{x^2 + y^2}\sin\frac{y}{x}$$

is homogeneous of first degree in x and y. Hence

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f,$$

as may be directly verified by calculating the partial derivatives and substituting in the equation.

Example 2. If $u = \dfrac{y}{x}$, $v = \dfrac{z^2}{x}$, $f(u, v) = 0$, we can consider u and v as homogeneous functions of zero degree, and z of degree $\frac{1}{2}$, in x and y. Thus

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z.$$

192. Properties of the Differential. The utility of the differential is due largely to the following properties, which we state for three variables, the extension to n variables being obvious.

(1) *The differential of $f(x, y, z)$ is*

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

whether x, y, z are the independent variables or not.

Suppose, for example, that there are two independent variables s, t and that x, y, z are functions of s and t. By the definition of differential we then have

$$dx = \frac{\partial x}{\partial s} \Delta s + \frac{\partial x}{\partial t} \Delta t,$$

$$dy = \frac{\partial y}{\partial s} \Delta s + \frac{\partial y}{\partial t} \Delta t,$$

$$dz = \frac{\partial z}{\partial s} \Delta s + \frac{\partial z}{\partial t} \Delta t,$$

$$df = \frac{\partial f}{\partial s} \Delta s + \frac{\partial f}{\partial t} \Delta t. \tag{1}$$

Also [§190, (2)]

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s},$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

These values being substituted, (1) becomes

$$\begin{aligned} df &= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}\right)\Delta s + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\right)\Delta t \\ &= \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial s}\Delta s + \frac{\partial x}{\partial t}\Delta t\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial s}\Delta s + \frac{\partial y}{\partial t}\Delta t\right) + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial s}\Delta s + \frac{\partial z}{\partial t}\Delta t\right) \\ &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz, \end{aligned}$$

which was to be proved.

(2) *If $f(x, y, z)$ is constant, $df = 0$.*

Suppose again that x, y, z are functions of two independent variables s and t. Then

$$df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt.$$

Since s and t can be varied independently and f does not change,

$$\frac{\partial f}{\partial s} = 0, \quad \frac{\partial f}{\partial t} = 0,$$

and hence $df = 0$.

(3) *A partial derivative* $\dfrac{\partial u}{\partial x}$ *is the ratio of two differentials* du *and* dx, *these differentials being determined with the same quantities constant as in the partial derivative.*

For example, to determine

$$\left(\frac{\partial u}{\partial x}\right)_{y,z}$$

we keep y and z constant in the equation

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz,$$

thus obtaining

$$du = \frac{\partial u}{\partial x}\,dx$$

and hence

$$\frac{\partial u}{\partial x} = \frac{du}{dx}.$$

When the variables satisfy given equations this often enables us to determine partial derivatives by algebraic elimination. The procedure is to differentiate the equations, leaving constant the same quantities as in the desired derivative, and then solve for the appropriate ratio of differentials.

Example 1. The equation $f(x, y, z) = 0$ determines z as an implicit function of x and y. Find $\dfrac{\partial z}{\partial x}$.

By differentiating

$$f(x, y, z) = 0$$

with y constant and z a function of x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0,$$

whence

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}.$$

Example 2. If $u = x^2 + y^2 + z^2$, $v = xy$, find $\left(\frac{\partial z}{\partial v}\right)_{u,y}$.

Differentiating the two equations with u and y constant, we get

$$0 = 2x\,dx + 2z\,dz,$$

$$dv = y\,dx.$$

Eliminating dx, we find

$$yz\,dz = -x\,dv.$$

Thus

$$\left(\frac{\partial z}{\partial v}\right)_{u,y} = -\frac{x}{yz}.$$

193. Tangent Plane and Normal to a Surface. If the point $P(x, y, z)$ describes a curve on the surface

$$F(x, y, z) = 0, \tag{1}$$

the differentials of its coördinates satisfy the equation

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0. \tag{2}$$

In particular, the differentials at the point $P_1(x_1, y_1, z_1)$ satisfy the equation

$$\left(\frac{\partial F}{\partial x}\right)_1 dx + \left(\frac{\partial F}{\partial y}\right)_1 dy + \left(\frac{\partial F}{\partial z}\right)_1 dz = 0, \tag{3}$$

$\left(\frac{\partial F}{\partial x}\right)_1, \left(\frac{\partial F}{\partial y}\right)_1, \left(\frac{\partial F}{\partial z}\right)_1$ being the values of the derivatives at P_1.

Along a given curve dx, dy, dz are proportional to the direction cosines of the tangent [§181, (4)]. Equation (3) thus states that the tangent at P_1 to any curve on the surface is perpendicular to the vector

$$\mathbf{N} = \mathbf{i}\left(\frac{\partial F}{\partial x}\right)_1 + \mathbf{j}\left(\frac{\partial F}{\partial y}\right)_1 + \mathbf{k}\left(\frac{\partial F}{\partial z}\right)_1. \tag{4}$$

All tangents at P_1, being perpendicular to this vector, lie in a plane called the *tangent plane* to the surface. Since it passes through $P_1(x_1, y_1, z_1)$ and is perpendicular to the above vector, the equation of the tangent plane is (§174)

$$\left(\frac{\partial F}{\partial x}\right)_1(x - x_1) + \left(\frac{\partial F}{\partial y}\right)_1(y - y_1) + \left(\frac{\partial F}{\partial z}\right)_1(z - z_1) = 0. \tag{5}$$

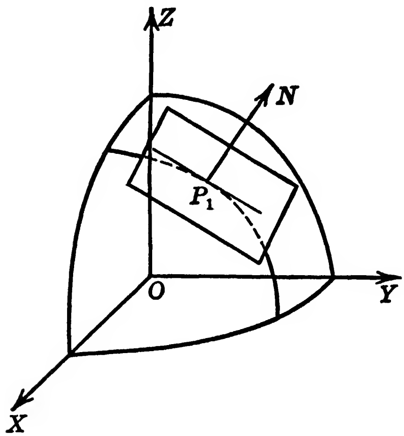

FIGURE 244.

A line perpendicular to the tangent plane at its point of contact is called *normal* to the surface. The above discussion shows that the vector **N** extends along the normal.

Example. Find the tangent plane and normal to the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 3x + 12$$

at (2, −1, 2).

The equation can be written

$$x^2 + 2y^2 + 3z^2 - 3x - 12 = 0.$$

The normal vector at (x, y, z) is therefore

$$\mathbf{i}(2x - 3) + \mathbf{j}(4y) + \mathbf{k}(6z).$$

At (2, −1, 2) this has the value

$$\mathbf{N} = \mathbf{i} - 4\mathbf{j} + 12\mathbf{k}.$$

A line through (2, −1, 2) and having the direction of this vector is normal to the surface. The equation of the tangent plane is

$$(x - 2) - 4(y + 1) + 12(z - 2) = 0.$$

194. Spherical Coördinates. The *spherical coördinates* of a point P in space are its distance $r = OP$ from the origin, the angle ϕ from OZ to OP, and the angle θ from the x-axis to the plane OPZ. These coördinates are particularly convenient when distance from the origin plays an important part.

The positive direction of θ is that of right-handed rotation about OZ. The positive direction of ϕ is that of rotation through

90° from OZ to the terminal side of θ. The positive direction of r is along the terminal side of ϕ.

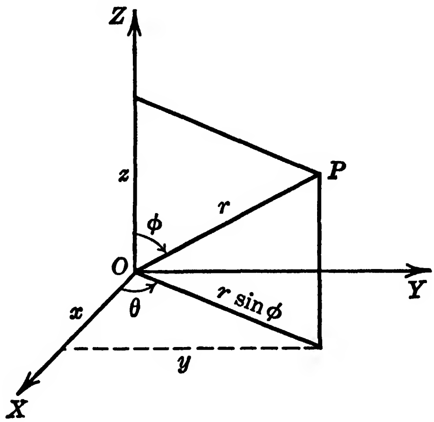

FIGURE 245.

The equations connecting the rectangular and spherical coördinates of the same point may be read from the diagram (Figure 245). The most important of these are

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi. \tag{1}$$

The locus of points r = constant is a sphere with center at the origin; ϕ = constant is a cone with axis OZ; θ = constant is a plane through OZ.

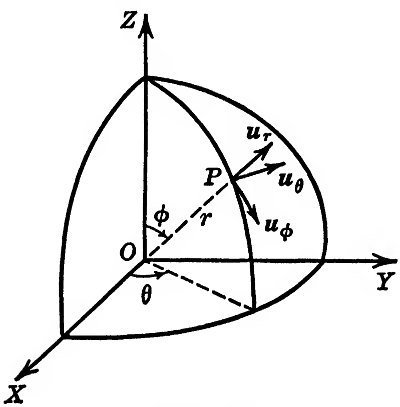

FIGURE 246.

At a point P with spherical coördinates r, ϕ, θ are three unit vectors $\mathbf{u}_r$, $\mathbf{u}_\phi$, $\mathbf{u}_\theta$ drawn in the directions P moves when two of the coördinates are kept constant and the third r, ϕ, or θ increases. When r alone varies these unit vectors do not change. They are therefore functions of ϕ and θ only.

When ϕ alone changes $\mathbf{u}_\theta$ does not change, and r, ϕ are plane polar coördinates. Thus

$$\frac{\partial \mathbf{u}_r}{\partial \phi} = \mathbf{u}_\phi, \quad \frac{\partial \mathbf{u}_\phi}{\partial \phi} = -\mathbf{u}_r, \quad \frac{\partial \mathbf{u}_\theta}{\partial \phi} = 0. \tag{2}$$

When ϕ remains constant and a change $d\theta$ is made in θ, the system of vectors rotates through the angle $d\theta$ about the z-axis. Thus (§183)

$$\frac{\partial \mathbf{u}_r}{\partial \theta} = \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta \sin \phi, \tag{3}$$

$$\frac{\partial \mathbf{u}_\phi}{\partial \theta} = \mathbf{k} \times \mathbf{u}_\phi = \mathbf{u}_\theta \cos \phi. \tag{4}$$

Finally, by differentiating $\mathbf{u}_\theta = \mathbf{u}_r \times \mathbf{u}_\phi$ and using the preceding equations, we get

$$\frac{\partial \mathbf{u}_\theta}{\partial \theta} = -\mathbf{u}_r \sin \phi - \mathbf{u}_\phi \cos \phi. \tag{5}$$

These values of the derivatives give for the differentials of the unit vectors

$$\left.\begin{aligned} d\mathbf{u}_r &= \mathbf{u}_\phi\, d\phi + \mathbf{u}_\theta \sin \phi\, d\theta, \\ d\mathbf{u}_\phi &= -\mathbf{u}_r\, d\phi + \mathbf{u}_\theta \cos \phi\, d\theta, \\ d\mathbf{u}_\theta &= -(\mathbf{u}_r \sin \phi + \mathbf{u}_\phi \cos \phi)\, d\theta. \end{aligned}\right\} \tag{6}$$

When spherical coördinates are used, components of vectors are taken along $\mathbf{u}_r$, $\mathbf{u}_\phi$, $\mathbf{u}_\theta$. Thus, if

$$\mathbf{r} = r\mathbf{u}_r \tag{7}$$

is the vector from the origin to the point P with coördinates r, ϕ, θ,

$$d\mathbf{r} = \mathbf{u}_r\, dr + \mathbf{u}_\phi r\, d\phi + \mathbf{u}_\theta r \sin \phi\, d\theta \tag{8}$$

is its differential, and $ds = |\, d\mathbf{r}\, |$ is the differential of arc. Since $\mathbf{u}_r$, $\mathbf{u}_\phi$, $\mathbf{u}_\theta$ are perpendicular unit vectors, this last has the value

$$ds = \sqrt{dr^2 + r^2\, d\phi^2 + r^2 \sin^2 \phi\, d\theta^2}. \tag{9}$$

If P is moving its velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{u}_r \frac{dr}{dt} + \mathbf{u}_\phi r \frac{d\phi}{dt} + \mathbf{u}_\theta r \sin \phi \frac{d\theta}{dt}. \tag{10}$$

The components of velocity are therefore

$$\frac{dr}{dt}, \quad r\frac{d\phi}{dt}, \quad r \sin \phi \frac{d\theta}{dt}.$$

By differentiating $\mathbf{v}$ and using equations (6) an expression for acceleration can be obtained.

195. Maxima and Minima of Functions of Two or More Variables. A maximum value of a function is a value greater than any given by neighboring values of the variables. A minimum value is a value less than any given by neighboring values.

If u is a function of any number of independent variables, it is clear that a maximum or minimum value of u remains a maximum or minimum when only one variable changes. *If then a maximum or minimum occurs at an interior point of a region in which all the first derivatives with respect to the independent variables are continuous, those derivatives must be zero.*

At a maximum or minimum on the boundary these conditions need not, however, be satisfied. On the section cut from a solid by an inclined plane, for example, the maximum value of z occurs at a boundary point and the partial derivatives of z with respect to x and y are not in general zero at that point.

When all the partial derivatives are zero, the total differential is zero. Thus, if u is a function of independent variables x, y, z, and all the partial derivatives with respect to those variables are zero,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

for all values of dx, dy, dz.

To find maxima and minima of a function of several variables, we equate its partial derivatives with respect to the *independent variables* (or its total differential) to zero, and solve the resulting equations. It is usually possible to decide from the problem whether a value thus found is a maximum, a minimum, or neither.

Example 1. The vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3). Find the point in the plane of the triangle such that the sum of the squares of its distances from the vertices is least.

Let (x, y) be the required point. The sum of the squares of the distances is

$$S = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + (x - x_3)^2 + (y - y_3)^2.$$

The variables x and y are independent. Thus

$$\frac{\partial S}{\partial x} = 2(x - x_1) + 2(x - x_2) + 2(x - x_3) = 0,$$

$$\frac{\partial S}{\partial y} = 2(y - y_1) + 2(y - y_2) + 2(y - y_3) = 0,$$

whence

$$x = \tfrac{1}{3}(x_1 + x_2 + x_3), \quad y = \tfrac{1}{3}(y_1 + y_2 + y_3).$$

The sum evidently has a least value. The partial derivatives must be zero at the minimum. The equations have only one solution. It must give the minimum.

Example 2. Find the point in the plane

$$x + 2y + 3z = 14$$

nearest the origin.

The distance from any point (x, y, z) of the plane to the origin is

$$D = \sqrt{x^2 + y^2 + z^2}.$$

If this is a minimum

$$dD = \frac{xdx + y\,dy + zdz}{\sqrt{x^2 + y^2 + z^2}} = 0,$$

that is

$$x\,dx + y\,dy + z\,dz = 0. \tag{1}$$

From the equation of the plane we get

$$dx + 2dy + 3dz = 0. \tag{2}$$

The differentials can have any values satisfying this equation. If these are all to satisfy (1), that equation must be a mere multiple of (2). Corresponding coefficients must therefore be proportional. Hence

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

Solving these and the equation of the plane simultaneously, we get $x = 1$, $y = 2$, $z = 3$. There is a minimum and it must satisfy the above equations. Since there is only one solution, it gives the minimum.

196. Higher Derivatives. The first partial derivatives are functions of the variables. By differentiating these partially we get higher derivatives.

For example, the first derivatives of $f(x, y)$ are

$$\frac{\partial f}{\partial x} = f_x(x, y), \quad \frac{\partial f}{\partial y} = f_y(x, y).$$

By differentiating these we obtain the second derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}(x, y),$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}(x, y),$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}(x, y),$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}(x, y).$$

By differentiating again, we get third derivatives, etc.

There is an order to partial differentiation, the operation indicated by the symbol on the left being performed last. Two partial derivatives of the nth order, which differ merely in the order of operations, are, however, equal provided that all partial derivatives of order equal to or less than n are continuous. This is a consequence of the following theorem:

At a point where $\dfrac{\partial^2 f}{\partial x\, \partial y}$ *and* $\dfrac{\partial^2 f}{\partial y\, \partial x}$ *are both continuous they are equal.*

Let $f_{xy}(x, y)$ and $f_{yx}(x, y)$ be continuous at (a, b). We are to prove

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{1}$$

For that purpose we use the second difference

$$\Delta = f(a + h, b + k) - f(a, b + k) - f(a + h, b) + f(a, b). \tag{2}$$

This can be written

$$\Delta = [f(a + h, b + k) - f(a, b + k)] - [f(a + h, b) - f(a, b)]. \tag{3}$$

The expression in the first bracket is obtained from that in the second by replacing b by $b + k$. From the mean value theorem (§34) we thus have

$$\Delta = k[f_y(a + h, b_1) - f_y(a, b_1)], \tag{4}$$

b_1 being between b and $b + k$. By a second application of the mean value theorem (4) becomes

$$\Delta = hkf_{xy}(a_1, b_1), \tag{5}$$

a_1 being between a and $a + h$.

Equation (2) can also be written

$$\Delta = [f(a + h, b + k) - f(a + h, b)] - [f(a, b + k) - f(a, b)]. \tag{6}$$

By a procedure similar to that above, but considering first the change from a to $a + h$, then that from b to $b + k$, we get

$$\Delta = khf_{yx}(a_2, b_2), \tag{7}$$

where a_2 is between a and $a + h$, and b_2 is between b and $b + k$.

From (5) and (7) we have

$$f_{xy}(a_1, b_1) = f_{yx}(a_2, b_2). \tag{8}$$

When h and k tend to zero, a_1 and a_2 tend to a, and b_1 and b_2 tend to b. Since f_{xy} and f_{yx} are continuous at (a, b), equation (8) thus gives as limit

$$f_{xy}(a, b) = f_{yx}(a, b),$$

which was to be proved.

Example 1. Assuming all derivatives of third or lower order continuous, prove that

$$\frac{\partial^3 f}{\partial x^2\,\partial y} = \frac{\partial^3 f}{\partial y\,\partial x^2}.$$

Under the conditions stated we have

$$\frac{\partial^3 f}{\partial x^2\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y\,\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^3 f}{\partial y\,\partial x^2}.$$

Example 2. If the substitution $x = r\cos\theta$, $y = r\sin\theta$ changes $f(x, y)$ into $F(r, \theta)$, express

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$$

in terms of derivatives of $f(x, y)$ with respect to x and y.

By assumption

$$F(r, \theta) = f(x, y).$$

Hence

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y},$$

$$\frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}.$$

By differentiating again and replacing the derivatives of x and y, we obtain

$$\frac{\partial^2 F}{\partial r^2} = \cos^2\theta\,\frac{\partial^2 f}{\partial x^2} + 2\cos\theta\sin\theta\,\frac{\partial^2 f}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 f}{\partial y^2},$$

$$\frac{\partial^2 F}{\partial \theta^2} = r^2\sin^2\theta\,\frac{\partial^2 f}{\partial x^2} - 2r^2\cos\theta\sin\theta\,\frac{\partial^2 f}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 f}{\partial y^2}$$

$$- r\cos\theta\,\frac{\partial f}{\partial x} - r\sin\theta\,\frac{\partial f}{\partial y}.$$

Substituting these values for the derivatives and combining terms, we obtain

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

197. Taylor's Series. The functions $f(x, y)$ most frequently used are for most values of a and b expressible as double series

$$f(x, y) = \sum_{m,n=0}^{\infty} A_{mn}(x-a)^m(y-b)^n \tag{1}$$

convergent when $|x - a|$ and $|y - b|$ are sufficiently small. Assuming such an expansion to exist and to be differentiable term by term, we find

$$A_{mn} = \frac{1}{m!n!}\left(\frac{\partial^{m+n}f}{\partial x^m\,\partial y^n}\right)_{(a,b)}, \tag{2}$$

the subscript (a, b) signifying that after differentiation x, y are to be replaced by a, b.

Unless the precise order of terms is specified (which is not usually the case), convergence of (1) can only mean absolute convergence. By a discussion similar to that in §148 we show that if the series converges at $x = x_1$, $y = y_1$ it converges absolutely for all values x, y such that

$$|x - a| < |x_1 - a|, \quad |y - b| < |y_1 - b|.$$

The region of convergence is therefore symmetric with respect to (a, b) but may still be very complicated.

If we write

$$x = a + ht, \quad y = b + kt \tag{3}$$

and consider h and k constant, $f(x, y)$ becomes a function $F(t)$ defined by the equation

$$F(t) = f(a + ht, b + kt). \tag{4}$$

Assuming sufficient differentiability, this function can be expanded by the Maclaurin formula

$$F(t) = F(0) + F'(0)t + \cdots + \frac{F^n(0)}{n!}t^n + R_n, \tag{5}$$

the remainder R_n being determined as in §150. Now

$$F'(t) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}.$$

Similarly,

$$F^n(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f, \tag{6}$$

the notation signifying that the expression

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$$

is to be expanded by the binomial theorem and applied to $f(x, y)$. Equation (5) can therefore be written

$$f(a + ht, b + kt) = \sum_{m=0}^{n} \frac{t^m}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f_0 + R_n, \tag{7}$$

the subscript zero signifying that after differentiation t is to be replaced by zero, that is, x and y are to be replaced by a and b.

In (7) t is arbitrary. Replacing t by 1, we have

$$f(a + h, b + k) = \sum_{m=0}^{n} \frac{1}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f_0 + R_n. \tag{8}$$

By a discussion similar to that in §150 the remainder can be shown to have the form

$$R_n = \frac{1}{(n + 1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f, \tag{9}$$

the derivatives being evaluated at some point between (a, b) and $(a + h, b + k)$, and on the joining line.

Equation (8) is called Taylor's formula with remainder. If the remainder tends to zero as n increases, it gives Taylor's series. Provided the partial derivatives used are continuous, there is an analogous expansion for a function of any number of variables.

198. Exact Differentials. The expression

$$M\,dx + N\,dy,$$

where M and N are functions of x and y, may or may not be the total differential of some function $f(x, y)$. If there exists a function $f(x, y)$ such that

$$df = M\,dx + N\,dy,$$

the expression $M\,dx + N\,dy$ is called an *exact differential.*

Likewise, if P, Q, R are functions of x, y, z and there exists a function $f(x, y, z)$ such that

$$df = P\,dx + Q\,dy + R\,dz,$$

the expression $P\,dx + Q\,dy + R\,dz$ is called an exact differential. A similar definition may be given for any number of variables.

If x and y are independent variables, a necessary and sufficient condition that

$$M\,dx + N\,dy$$

be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{1}$$

To prove this suppose first that there is a function $f(x, y)$ such that

$$df = M\,dx + N\,dy.$$

Since

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

for all values of dx and dy, we must then have

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N,$$

and hence

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}.$$

The two second derivatives being equal, this requires

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

which was to be proved.

Next suppose, conversely, that the condition (1) is satisfied. Let

$$w = \int M\,dx$$

be the function obtained by integrating with y constant. The expression

$$N - \frac{\partial w}{\partial y}$$

is a function of y only; for

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial M}{\partial y},$$

and hence

$$\frac{\partial}{\partial x}\left(N - \frac{\partial w}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Let

$$f(x, y) = w + \int\left(N - \frac{\partial w}{\partial y}\right) dy.$$

Since the second term on the right is a function of y only, we have

$$\begin{aligned} df &= \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \left(N - \frac{\partial w}{\partial y}\right) dy \\ &= M\,dx + N\,dy, \end{aligned}$$

which shows that $M\,dx + N\,dy$ is exact.

In a similar way we prove that

If x, y, z are independent variables, a necessary and sufficient condition that

$$P\,dx + Q\,dy + R\,dz$$

be an exact differential is that P, Q, R satisfy the equations

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}. \tag{2}$$

Example 1. Show that

$$(2x - 3y)\,dx - (3x + 2y)\,dy$$

is an exact differential.

In this case

$$M = 2x - 3y, \quad N = -(3x + 2y)$$

$$\frac{\partial M}{\partial y} = -3, \quad \frac{\partial N}{\partial x} = -3.$$

The two partial derivatives being equal, the expression is exact.

Example 2. In thermodynamics it is shown that

$$dU = T\,dS - p\,dv,$$

U being the internal energy, T the absolute temperature, S the entropy, p the pressure, and v the volume of a homogeneous substance. Any two

of these quantities can be assigned independently, and the others are then determined. Show that

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial v}{\partial S}\right)_p.$$

The result to be proved expresses that

$$T\,dS + v\,dp$$

is an exact differential. That such is the case is shown by replacing $T\,dS$ by its value $dU + p\,dv$. We thus get

$$T\,dS + v\,dp = dU + p\,dv + v\,dp = d(U + pv).$$

199. Directional Derivative. Gradient. Let $f(x, y, z)$ be a function with first derivatives continuous at points $P(x, y, z)$ of a certain region. When P moves along a given direction, $f(x, y, z)$ may be considered a function of the distance s moved. The derivative of f with respect to s,

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}, \tag{1}$$

is called the *derivative of $f(x, y, z)$ in the assigned direction.* It is a function of the position $P(x, y, z)$ and the direction of motion.

In particular, if P moves parallel to the x-axis in the positive direction, $dy = dz = 0$, $ds = dx$, and

$$\frac{df}{ds} = \frac{\partial f}{\partial x}.$$

The partial derivative of f with respect to a coördinate is thus merely the derivative of f in the positive direction parallel to the corresponding axis.

At a point where $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ are continuous, the vector with components equal to these derivatives is called the *gradient* of f and is represented by the notation

$$\operatorname{grad} f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}. \tag{2}$$

To see the significance of this let

$$\mathbf{u} = \mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{ds} + \mathbf{k}\frac{dz}{ds} \tag{3}$$

be the unit vector in a given direction. Equation (1) can then be written

$$\frac{df}{ds} = \mathbf{u}\cdot\operatorname{grad} f. \tag{4}$$

This states that the derivative of a function in any direction is equal to the component of its gradient in that direction.

If components of a vector are taken along different directions, the component will be greatest along the direction of the vector itself. *The gradient of $f(x, y, z)$ therefore has the magnitude and direction of the greatest directional derivative of $f(x, y, z)$ at (x, y, z).*

Suppose, for example, that $T(x, y, z)$ is the temperature at the point $P(x, y, z)$ of a region. As we start out from P in different directions there will be some direction in which the temperature increases most rapidly. The gradient of T points in that direction and is equal in magnitude to the most rapid increase per unit distance moved.

Through the point $P(x, y, z)$ in the region there passes a surface $f(x, y, z) = \text{constant}$. In a motion along this surface the component of grad f is

$$\frac{df}{ds} = 0,$$

and so the direction of motion is perpendicular to the gradient. Therefore, *the gradient of $f(x, y, z)$ is normal to the surface*

$$f(x, y, z) = \text{constant}$$

which passes through (x, y, z).

If $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is the vector from the origin to $P(x, y, z)$,

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \tag{5}$$

and

$$d\mathbf{r}\cdot\operatorname{grad} f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df. \tag{6}$$

If then

$$\mathbf{V} = \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z \tag{7}$$

is the gradient of a scalar function $f(x, y, z)$,

$$d\mathbf{r}\cdot\mathbf{V} = d\mathbf{r}\cdot\operatorname{grad} f = df \tag{8}$$

is an exact differential. Conversely, if

$$d\mathbf{r}\cdot\mathbf{V} = X\,dx + Y\,dy + Z\,dz$$

is the differential of a function $f(x, y, z)$, then

$$d\mathbf{r}\cdot\mathbf{V} = df = d\mathbf{r}\cdot\operatorname{grad} f \tag{9}$$

for all values of $d\mathbf{r}$, and consequently

$$\mathbf{V} = \operatorname{grad} f. \tag{10}$$

Therefore, *a necessary and sufficient condition that*

$$X\,dx + Y\,dy + Z\,dz$$

be an exact differential is that the vector

$$\mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z$$

be the gradient of some scalar function.

Equation (6) can be used to determine an expression for gradient when $d\mathbf{r}$ is known in a particular system of coördinates. Thus in spherical coördinates

$$d\mathbf{r} = \mathbf{u}_r\,dr + \mathbf{u}_\phi r\,d\phi + \mathbf{u}_\theta r\sin\phi\,d\theta.$$

To make the scalar product of this and grad f equal to

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial \theta}d\theta$$

we must have

$$\operatorname{grad} f = \mathbf{u}_r\frac{\partial f}{\partial r} + \frac{\mathbf{u}_\phi}{r}\frac{\partial f}{\partial \phi} + \frac{\mathbf{u}_\theta}{r\sin\phi}\frac{\partial f}{\partial \theta}. \tag{11}$$

Example 1. Find the derivative of $x^2 + y^2 + z^2$ in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at the point $(2, 3, -4)$.

In the assigned direction

$$\frac{dx}{ds} = \frac{1}{3},\quad \frac{dy}{ds} = \frac{2}{3},\quad \frac{dz}{ds} = \frac{2}{3}.$$

By equation (1) the derivative of $f = x^2 + y^2 + z^2$ in that direction is

$$2x\frac{dx}{ds} + 2y\frac{dy}{ds} + 2z\frac{dz}{ds} = 4\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right) - 8\left(\frac{2}{3}\right) = 0$$

at the point $(2, 3, -4)$.

Example 2. Find a vector normal to the surface $xyz = 6$ at the point $(1, 2, 3)$.

The gradient

$$\mathbf{i}\frac{\partial}{\partial x}(xyz) + \mathbf{j}\frac{\partial}{\partial y}(xyz) + \mathbf{k}\frac{\partial}{\partial z}(xyz) = \mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

is a vector normal to the surface at the given point.

200. Line Integral. Work. Suppose that a point $P(x, y, z)$ moves along a given curve under the action of a force

$$\mathbf{F} = \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z \tag{1}$$

with components that are in general functions of x, y, z. To estimate the work done by this force we replace the curve by a polygonal line with vertices on the curve and represent P by the vector

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{2}$$

from the origin to the point P. When the consecutive vertices $P(x, y, z)$, $Q(x + \Delta x, y + \Delta y, z + \Delta z)$ are sufficiently close

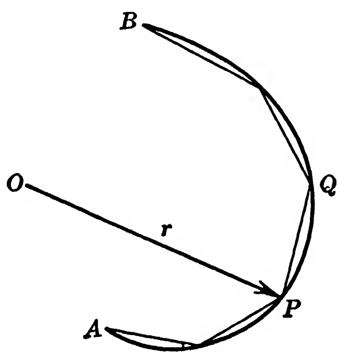

FIGURE 247.

together the force has almost the value $\mathbf{F}(x, y, z)$ at all points of the chord PQ. The work in describing this chord is therefore approximately

$$\mathbf{F}\cdot\overrightarrow{PQ} = \mathbf{F}\cdot\Delta\mathbf{r}$$

and the total work is approximately

$$\sum_{A}^{B}\mathbf{F}\cdot\Delta\mathbf{r}.$$

The work in passing along the curve from A to B is thus defined as the limit

$$\lim_{\Delta\mathbf{r}\to 0}\sum_{A}^{B}\mathbf{F}\cdot\Delta\mathbf{r} = \int_{A}^{B}\mathbf{F}\cdot d\mathbf{r}. \tag{3}$$

Substituting for $\mathbf{F}$ and $d\mathbf{r}$ from (1) and (2) the expression for work becomes

$$\int_A^B X\,dx + Y\,dy + Z\,dz. \tag{4}$$

An integral of the form (3) or (4) is called a *line integral.* To evaluate such an integral along a curve with continuously varying tangent we use the equations of the curve to express $\mathbf{F}$ and $d\mathbf{r}$, or x, y, z, X, Y, Z, in terms of a single variable t and its differential. The integral then takes the form

$$\int_{t_1}^{t_2} f(t)\,dt,$$

t_1 and t_2 being the values of t at A and B. When the curve is formed by sections with an abrupt change of direction at each junction, the integral is the sum of integrals along the separate sections.

Example 1. Find the work done by the force

$$\mathbf{F} = \mathbf{i}y - \mathbf{j}x$$

when the point of application moves in the xy-plane along the parabola $y^2 = 4x$ from (0, 0) to (1, 2).

In this case $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$ and

$$\int \mathbf{F}\cdot d\mathbf{r} = \int y\,dx - x\,dy.$$

Along the given curve $x = \frac{1}{4}y^2$ and

$$y\,dx - x\,dy = \tfrac{1}{4}y^2\,dy.$$

As the point moves along the curve, y varies from 0 to 2. Thus the work done is

$$\int \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \tfrac{1}{4}y^2\,dy = \tfrac{2}{3}.$$

Example 2. Find the value of the integral

$$\int x\,dy + y\,dz + z\,dx$$

along the curve

$$x = t, \quad y = 2t, \quad z = t^2$$

from the point where $t = -3$ to the point where $t = 3$.

Substituting for x, y, z the values from the equations of the curve, we get

$$x\,dy + y\,dz + z\,dx = (2t + 5t^2)\,dt.$$

The integral from $t = -3$ to $t = 3$ is therefore

$$\int_{-3}^{3}(2t + 5t^2)\,dt = [t^2 + \tfrac{5}{3}t^3]_{-3}^{3} = 90.$$

201. Case of Line Integral Independent of the Path. In general the line integral

$$\int_A^B \mathbf{F}\cdot d\mathbf{r}$$

depends on the curve from A to B along which it is taken. If two different paths join the same end points, the integral along one need not be the same as that along the other.

If, however, $\mathbf{F}$ *is the gradient of a one-valued function, the line integral depends only on the end points of the path.*

Suppose, in fact,

$$\mathbf{F} = \operatorname{grad}\phi \tag{1}$$

where $\phi(x, y, z)$ is one-valued in a certain region. For points in that region [§199, (6)]

$$\mathbf{F}\cdot d\mathbf{r} = d\mathbf{r}\cdot\operatorname{grad}\phi = d\phi$$

and consequently

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B d\phi = \phi(B) - \phi(A), \tag{2}$$

the notation $\phi(A)$ being used to represent the value of $\phi(x, y, z)$ at the point A, and $\phi(B)$ the value at B. The integral thus depends only on the end points A, B and not on the path connecting them.

When the integral is independent of the path, the integral around a closed path is zero. For, if A, B are any two points on a closed path the integral around the circuit can then be written

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} + \int_B^A \mathbf{F}\cdot d\mathbf{r} = \int_A^B \mathbf{F}\cdot d\mathbf{r} - \int_A^B \mathbf{F}\cdot d\mathbf{r} = 0. \tag{3}$$

Conversely, *if* $\mathbf{F}(x, y, z)$ *is continuous and the integral*

$$\int \mathbf{F}\cdot d\mathbf{r}$$

around every closed path in a three-dimensional region is zero, there is a function $\phi(x, y, z)$ *such that*

$$\mathbf{F} = \text{grad}\,\phi.$$

To show this let

$$\phi = \int_{P_0}^{P} \mathbf{F}\cdot d\mathbf{r}, \tag{4}$$

where P_0 is a fixed point and $P(x, y, z)$ a variable point in the region. Since the integral around every closed path is zero, the integral from P_0 to P does not depend on the path. For *any* integral from P_0 to P added to a particular integral from P to P_0 must give zero. Thus ϕ is a one-valued function of x, y, z. Take a particular path from P_0 to P along which the unit tangent vector

$$\mathbf{t} = \frac{d\mathbf{r}}{ds}$$

is continuous. Along this path

$$\phi = \int_{P_0}^{P} \mathbf{F}\cdot d\mathbf{r} = \int_{P_0}^{P} \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\,ds$$

is a function of s with derivative (§51)

$$\frac{d\phi}{ds} = \mathbf{F}\cdot\frac{d\mathbf{r}}{ds} \tag{5}$$

at P. The curve could be chosen to have any direction at P. Hence the derivative in any direction is given by (5). This shows that $\phi(x, y, z)$ has a derivative in each direction and that the derivative in any particular direction is a continuous function of x, y, z. Thus $\phi(x, y, z)$ has a gradient at P and consequently

$$\frac{d\phi}{ds} = \frac{d\mathbf{r}}{ds}\cdot\text{grad}\,\phi. \tag{6}$$

Combining this with (5) we have

$$(\mathbf{F} - \text{grad}\,\phi)\cdot\frac{d\mathbf{r}}{ds} = 0.$$

Since this holds for every value of $\dfrac{d\mathbf{r}}{ds}$,

$$\mathbf{F} - \text{grad}\,\phi = 0,$$

which was to be shown.

When the work done by a force is zero around every closed path, the force is called *conservative*. We have thus shown:

The necessary and sufficient condition for a force $F(x, y, z)$ to be conservative is that it be the gradient of a scalar function.

A conservative force is usually expressed as a negative gradient

$$\mathbf{F} = -\text{grad}\,\phi.$$

The function ϕ is then called a *potential*. For example, the force of gravity on a weight of w pounds is of magnitude w and directed downward. Taking the z-axis upward,

$$\mathbf{F} = -\mathbf{k}w = -\text{grad}\,(wz).$$

The force $\mathbf{F}$ is thus derived from a potential wz.

PROBLEMS

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in each of the following problems:

1. $z = x^2 - 2xy + 4y^2$.

2. $z = \sin(xy)$.

3. $z = \ln \dfrac{y}{x}$.

4. $z = \dfrac{x + y}{x - y}$.

In the following problems perform the indicated differentiations, x, y, z being the independent variables:

5. $\dfrac{\partial}{\partial x}(x^2 - 3xy + y^2 + 2xz)$.

6. $\dfrac{\partial}{\partial y}\sin(2x + 3y + z)$.

7. $\dfrac{\partial}{\partial z}\ln(xy^2z^3)$.

8. $\dfrac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2}$.

9. If $f(x, y, z) = x^3 + 3xy^2 + 3yz^2$, find $f_x(x, y, z) - f_y(x, y, z)$.

10. If $f(x, y) = x \ln \dfrac{y^2}{x}$, find the value of $f_x(4, 2) + f_y(3, 1)$.

11. If $z = \ln(x^2 + xy + y^2)$, find the value of

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}.$$

12. If $f(x, y) = (x + a)(y + b)$, a and b being constants, show that

$$f_x(x, y)f_y(x, y) = f(x, y).$$

13. If $x = u + v$, $y = u - v$, show that

$$\left(\frac{\partial y}{\partial x}\right)_u = -\left(\frac{\partial y}{\partial x}\right)_v.$$

14. If $x = r\cos\theta$, $y = r\sin\theta$, find

$$\left(\frac{\partial x}{\partial r}\right)_\theta\left(\frac{\partial x}{\partial r}\right)_y.$$

15. If $x = r\cos\theta$, $y = r\sin\theta$, find

$$\left(\frac{\partial x}{\partial \theta}\right)_y \left(\frac{\partial x}{\partial \theta}\right)_r .$$

16. Let a, b be the sides of a right triangle, c the hypotenuse, and p the perpendicular from the vertex of the right angle to the hypotenuse. Find

$$\left(\frac{\partial p}{\partial a}\right)_b .$$

17. If K is the area of a triangle, a side and two adjacent angles of which are c, A, B, determine

$$\left(\frac{\partial K}{\partial A}\right)_{c,B} .$$

18. The sides of a triangle are a, b, c, and the opposite angles are A, B, C. Find

$$\left(\frac{\partial a}{\partial A}\right)_{b,c} .$$

19. Find the lowest point on the surface

$$z = x^2 + y^2 - xy - 4x + 5y + 10.$$

20. Find the highest point on the surface

$$z = 6y - 2x - x^2 - y^2.$$

21. Show that the partial derivatives of $z = xy$ are zero at the origin but that the origin is not a high or a low point on the surface.

Find the total differential of each of the following functions:

22. $z = x^2 - xy + 3y^2$.

23. $r = \sqrt{x^2 + y^2 + z^2}$.

24. $u = x^2y^3z^4$.

25. $u = \ln(xyz)$.

26. Calculate dz and Δz if $z = y^2 - x^2 - xy$, $x = 2$, $y = 3$, $dx = \Delta x = 0.01$, and $dy = \Delta y = -0.02$.

27. Calculate du and Δu if $u = xyz$, $x = 1$, $y = 2$, $z = 3$, $dx = \Delta x = 0.1$, $dy = \Delta y = -0.1$, $dz = \Delta z = 0.2$.

28. A box is 12 inches long, 8 inches wide, and 6 inches deep. Considering the edges as independent variables, find the increment and the differential of the volume when each edge is increased $\frac{1}{4}$ inch.

29. If V is the volume of a cylinder of radius r and altitude h and $dr = t$, $dh = 2t$, show that

$$dV = At,$$

where A is the total surface area of the cylinder, including the two ends.

30. The hypotenuse and one side of a right triangle are found by measurement to be $c = 5$ feet, $b = 4$ feet, and the other side a is calculated. Find the maximum error in a due to errors of 0.01 foot in the measurements of c and b.

31. The acute angle A in the triangle of the preceding problem is calculated from the data given. Find the maximum error in A.

32. The distance from the origin to the point $P(6, 3, 2)$ is calculated by the formula

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Find the maximum error in r due to errors of 1% in x, y, z.

33. The volume of a cone is computed by the formula

$$V = \tfrac{1}{3}\pi r^2 h.$$

Find the percentage error in the volume if the measurement of the radius is $\frac{1}{2}$% too large and that of the altitude 1% too small.

34. Two sides and the included angle of a triangle are found to be $c = 50$ feet, $b = 20$ feet, $A = 60°$, and the third side a is calculated. Find the maximum error in a due to errors of 0.1 foot in the measurements of b and c, and 0.01 radian in the measurement of A.

In each of the following problems find the derivative of $f(x, y)$ with respect to t by formulas (2) or (3), §190, and check by expressing $f(x, y)$ in terms of t and then differentiating:

35. $f(x, y) = x^2y,\ x = t^2,\ y = t^3.$

36. $f(x, y) = \sqrt{x^2 + y^2},\ x = 3t,\ y = 4t.$

37. $f(x, y) = x^2 + xy,\ x = s + t,\ y = s - t.$

38. $f(x, y) = \ln(x - y),\ x = \frac{1}{2}(e^t + e^{-t}),\ y = \frac{1}{2}(e^t - e^{-t}).$

In each of the following problems find $\dfrac{\partial u}{\partial x}$, assuming that x and y are independent variables and that z is a function of x and y:

39. $u = z \sin(x + y).$

40. $u = x^2 - y^2 + xz - z^2.$

41. $u = ye^{x-z}.$

42. $u = \ln(y + z).$

43. If $u = f(x, y)$, $y = x$, find $\dfrac{du}{dx}$.

44. If $u = f(x, y)$, $y = 2x - s$, find $\left(\dfrac{\partial u}{\partial x}\right)_s$.

45. If $u = f(x, y, z)$, $z = x^2$, find $\left(\dfrac{\partial u}{\partial x}\right)_y$.

46. Find $\dfrac{du}{dx}$ if $u = F(x, y, z)$, $z = f(x, y)$, $y = \phi(x)$. Check by using the functions $u = xyz$, $z = xy$, $y = x^4$.

47. The substitution $y = x + v$ changes $f(x, y)$ into $F(x, v)$. Determine $\dfrac{\partial f}{\partial x}$ in terms of derivatives of $F(x, v)$ with respect to x and v. Check by using the function $f(x, y) = xy$.

48. If the substitution $x = r\cos\theta$, $y = r\sin\theta$ changes $f(x, y)$ into $F(r, \theta)$, show that

$$\left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

49. If the substitution $x = r\cosh\theta$, $y = r\sinh\theta$ changes $f(x, y)$ into $F(r, \theta)$, show that

$$\left(\frac{\partial F}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial F}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2.$$

50. If $u = f(x^2 + y^2)$, show that

$$y\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}.$$

51. If $u = f(ax + by)$, show that

$$b\frac{\partial u}{\partial x} = a\frac{\partial u}{\partial y}.$$

52. If $u = f\left(\frac{y}{x}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

53. If $u = f(\sin x + \sin y)$, show that

$$\sec x\frac{\partial u}{\partial x} = \sec y\frac{\partial u}{\partial y}.$$

54. If $u = \dfrac{x^2 + y^2}{x + y}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

55. If $z = f\left(\dfrac{x - y}{y}\right)$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

56. If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

57. If z is determined by solving the equation

$$f\left(\frac{y}{x}, \frac{z}{x}\right) = 0,$$

show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

58. If $f(x, y) = 0$, express $\dfrac{dy}{dx}$ in terms of derivatives of $f(x, y)$ with respect to x and y.

59. Let α, β, γ be the direction angles of a line, and consider γ as a function of α and β. Find the value of $\dfrac{\partial \gamma}{\partial \alpha}$ when $\alpha = 45°$ and $\beta = \gamma = 60°$.

60. If $f(x, y, z) = 0$, $z = x + y$, find $\dfrac{dz}{dx}$.

61. If $F(x, y, z) = 0$, show that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

62. If $x = u^2 - v^2$, $y = uv$, find

$$\left(\frac{\partial v}{\partial y}\right)_x.$$

63. If $x + y = uv$, $xy = u - v$, find

$$\left(\frac{\partial x}{\partial u}\right)_y.$$

64. If $u = x + y + z$, $v = xyz$, find

$$\left(\frac{\partial z}{\partial u}\right)_{v,y}.$$

65. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, find

$$\left(\frac{\partial u}{\partial x}\right)_{v,z}.$$

66. If $x = r\cos\theta$, $y = r\sin\theta$, show that

$$x\,dy - y\,dx = r^2\,d\theta, \quad dx^2 + dy^2 = dr^2 + r^2\,d\theta^2.$$

67. If $x = r\cosh\theta$, $y = r\sinh\theta$, show that

$$x\,dy - y\,dx = r^2\,d\theta, \quad dx^2 - dy^2 = dr^2 - r^2\,d\theta^2.$$

Find the equations of the tangent plane and the normal to each of the following surfaces at the point indicated.

68. $x^2 + y^2 + z^2 = 9$; $(1, 2, 2)$.

69. $x^2 - y^2 = 8$; $(3, -1, 1)$.

70. $z^2 = x^2 + y^2$; $(3, 4, 5)$.

71. $x^2 + y^2 = 2z$; $(1, 3, 5)$.

72. $z = xy$; $(2, -3, -6)$.

73. $z^2 - x^2 - y^2 = 1$; $(2, -2, 3)$.

74. Find the components of acceleration in spherical coördinates.

75. A point moves on the sphere $r = a$. Find the components of its acceleration in spherical coördinates.

76. If the equations of a curve are given in spherical coördinates, show that

$$\mathbf{t} = \mathbf{u}_r \frac{dr}{ds} + \mathbf{u}_\phi r \frac{d\phi}{ds} + \mathbf{u}_\theta \sin\phi \frac{d\theta}{ds}$$

is the unit vector tangent to the curve.

77. In any motion of the point P (Figure 246) show that

$$\frac{d\mathbf{u}_r}{dt} = \boldsymbol{\omega} \times \mathbf{u}_r, \quad \frac{d\mathbf{u}_\theta}{dt} = \boldsymbol{\omega} \times \mathbf{u}_\theta, \quad \frac{d\mathbf{u}_\phi}{dt} = \boldsymbol{\omega} \times \mathbf{u}_\phi,$$

where

$$\boldsymbol{\omega} = \mathbf{u}_\theta \frac{d\phi}{dt} + \mathbf{k}\frac{d\theta}{dt}.$$

The unit vectors thus rotate with angular velocity $\boldsymbol{\omega}$.

78. Find the angular acceleration $\frac{d\omega}{dt}$ in the preceding problem.

79. Find the points on the surface $z^2 = xy + 1$ at least distance from the origin.

80. A tent having the form of a cylinder surmounted by a cone is to contain a given volume. Find its shape if the canvas required is a minimum.

81. When an electric current of strength I flows through a wire of resistance R the heat produced is proportional to I^2R. Two terminals are connected by three wires in parallel of resistances R_1, R_2, R_3, respectively. A given current flowing between the terminals will divide in such a way that the total heat produced is a minimum. Show that the currents I_1, I_2, I_3 in the three wires will satisfy the equations

$$I_1R_1 = I_2R_2 = I_3R_3.$$

82. Two adjacent sides of a room are plane mirrors. A ray of light starting from a lamp A in the ceiling strikes one of the mirrors at P, is reflected to a point Q on the second mirror, and is then reflected to a point B on the floor. Determine the positions of P and Q so that the path $APQB$ shall be as short as possible. Show that the angles of incidence and reflection are equal at each mirror.

83. Find the point on the curve

$$x^2 + y^2 = 2z, \quad x + y - z + 3 = 0$$

at greatest distance from the xy-plane.

In each of the following problems calculate $\frac{\partial^2 z}{\partial x\, \partial y}$ and $\frac{\partial^2 z}{\partial y\, \partial x}$, and show that they are equal:

84. $z = x^3 - x^2y + xy^2 - y^3$.

85. $z = x \ln (x + y)$.

86. $z = \frac{x - y}{x + y}$.

87. $z = x^y$.

88. Show that

$$\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial z} - z\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(z\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}\right) = 0,$$

x, y, z being the independent variables.

89. If $f(x, y) = 0$, find $\frac{d^2y}{dx^2}$.

90. If $z = f(x, y)$, $y = x^2$, find $\frac{d^2z}{dx^2}$.

91. If the transformation $x = e^u \cos v$, $y = e^u \sin v$ changes $f(x, y)$ into $F(u\ v)$, express

$$\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2}$$

in terms of derivatives of $f(x, y)$ with respect to x and y.

92. If the substitution $u = \frac{1}{2}(x^2 + y^2)$, $v = xy$ changes $F(u, v)$ into $f(x, y)$, express

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

in terms of derivatives of $F(u, v)$ with respect to u and v.

93. If the substitution $x = u + v$, $y = u - v$ changes $f(x, y)$ into $F(u, v)$, express

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}$$

in terms of derivatives of $F(u, v)$ with respect to u and v.

94. If the substitution $x = r \cosh \theta$, $y = r \sinh \theta$ changes $f(x, y)$ into $F(r, \theta)$, express

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} - \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$$

in terms of derivatives of $f(x, y)$ with respect to x and y.

95. Determine the coefficients A_{mn} in the series

$$\frac{1}{\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)} = \Sigma A_{mn} x^m y^n.$$

96. Determine the coefficients A_{mn} in the series

$$\frac{1}{1 - \left(\frac{x}{a} + \frac{y}{b}\right)} = \Sigma A_{mn} x^m y^n.$$

Determine which of the following are exact differentials:

97. $(2x + y)\,dx - (x - 3y)\,dy$.
98. $(x^2 - y^2)\,dx + (y^2 - 2xy)\,dy$.
99. $x\,dx - (y + z)\,dy + y\,dz$.
100. $(x^2 + y^2)\,dx - 2x\,dy$.
101. $(z + x)\,dx - (y + z)\,dy + (x - y)\,dz$.
102. $(\sin y - y \sin x + x)\,dx + (\cos x + x \cos y + y)\,dy$.
103. $(2xe^y + y)\,dx + (x^2e^y + x - 2y)\,dy$.

104. Find the derivative of $xy + yz$ in the direction of the vector

$$2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

at the point (1, 2, 2).

105. Find the derivative of $x + y + z$ in the direction of the vector

$$\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

106. Find the derivative of $f(x, y, z)$ in the direction $-\mathbf{i}$.

107. Find the gradient of $(x + y)e^z$.

108. If $f(x, y, z) = xy + yz + zx$, find the maximum value of $\dfrac{df}{ds}$ at the point $(2, -1, 3)$.

109. If θ is the angle in cylindrical coördinates find the gradient of θ.

110. Find the work done by the force

$$\mathbf{F} = \mathbf{i}(x^2 + y^2) - 2\mathbf{j}xy$$

when the point of application moves along the straight line $y = 2x$, $z = 3x$ from the origin to the point (1, 2, 3).

111. Find the work done by the force

$$\mathbf{F} = 5\mathbf{j}z - 4\mathbf{i}xy$$

when the point of application moves along the curve

$$y = 2x^2, \quad z = 2x^3$$

from $x = 0$ to $x = 2$.

112. Find the work done by the force $\mathbf{F} = \mathbf{j}xy^2$ when the point of application moves in the positive direction around the square formed by the lines $x = \pm 1$, $y = \pm 1$ in the xy-plane.

Find the value of each of the following integrals along the path given:

113. $\int y\,dx - z\,dy - x\,dz;\quad x = t,\ y = -t^2,\ z = 3t,\ 0 \leqq t \leqq 3.$

114. $\int \dfrac{x\,dz - z\,dx}{x^2 + y^2 + z^2};\quad x = 4t,\ y = 3t,\ z = 5t^2,\ -1 \leqq t \leqq 1.$

115. $\int y \sin x\,dx - \cos x\,dy + dz;\quad y = \cos x,\ z = \sin 2x,\ 0 \leqq x \leqq \dfrac{\pi}{2}.$

116. If $\mathbf{F} = \mathbf{i}y$, show that

$$\int_A^B \mathbf{F}\cdot d\mathbf{r}$$

along an arc AB in the xy-plane is equal to the area bounded by the x-axis, the arc, and the ordinates at A and B.

117. If $\mathbf{F} = \mathbf{i}x^2$, show that

$$\int_A^B \mathbf{F}\cdot d\mathbf{r}$$

is independent of the path from A to B.

118. If w is constant, show that $\mathbf{F} = \mathbf{k}w$ is a conservative force.

119. If $\mathbf{r}$ is the vector and r the distance from the origin to $P(x, y, z)$, show that $\mathbf{F} = r\mathbf{r}$ is a conservative force.

120. For paths in the xy-plane that do not go around the origin show that

$$\mathbf{F} = \frac{\mathbf{i}y - \mathbf{j}x}{x^2 + y^2}$$

is a conservative force.

CHAPTER XVII

MULTIPLE INTEGRATION

202. Repeated Integration. We found in §46 that, if $A(x)$ is the area of section cut from a solid by the plane $x =$ constant,

$$V = \int_a^b A(x)\,dx \tag{1}$$

is the volume of the solid between the planes $x = a$, $x = b$. When the area of section is known as a function of x we can therefore determine the volume by integration.

If, however, the section has curved boundaries (as it usually does), the determination of its area requires integration. In the section $x =$ constant of Figure 248, for example, y and z are plane

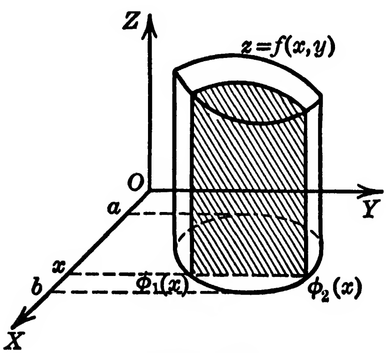

FIGURE 248.

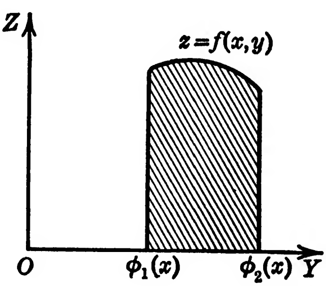

FIGURE 249.

coördinates and the section extends from $z = 0$ to the top surface $z = f(x, y)$. The area of section is therefore

$$A(x) = \int_{\phi_1(x)}^{\phi_2(x)} z\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy, \tag{2}$$

$\phi_1(x)$ and $\phi_2(x)$ being the values of y at the boundaries of the section.

To determine the volume we thus integrate (2) *with x constant*, substitute the resulting function of x in (1), and integrate again. The two integrations are indicated by the notation

$$V = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy \right) dx \tag{3}$$

and the result is called a *repeated integral*. The first integration is with respect to y with x constant between the limits $\phi_1(x)$ and $\phi_2(x)$. The second integration is with respect to x between a and b.

It should be noted that there is a definite order to these operations. In (3) the order is indicated by parentheses, the inner integration being performed first. The parentheses are, however, usually omitted, and instead of (3) we write

$$V = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx, \tag{4}$$

the symbols being arranged in the order they would have if parentheses were used. The variable in the first integration is thus indicated by the differential on the left, and its limits are attached to the integral sign on the right. It should be noted, however, that this usage is not universal. Some writers arrange the symbols differently.

Since $f(x, y)$ is the value of z we could write

$$f(x, y) = \int_0^{f(x, y)} dz$$

and so determine the volume

$$V = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_0^{f(x, y)} dz\, dy\, dx \tag{5}$$

by three integrations. In the present instance this is an unnecessary complication, but we shall encounter cases, moment of inertia, for example, in which all three integrations are needed.

Usually the limits in the earlier integrations are functions of the later variables, only the last limits being constant.

Example 1. Find the value of the integral

$$\int_0^3 \int_0^y (x^2 + y^2)\, dx\, dy.$$

Integrating with respect to x between the limits 0 and y, and then with respect to y, we obtain

$$\int_0^3 \int_0^y (x^2 + y^2)\, dx\, dy = \int_0^3 \left[\frac{x^3}{3} + xy^2\right]_0^y dy$$
$$= \int_0^3 \tfrac{4}{3}y^3\, dy = [\tfrac{1}{3}y^4]_0^3 = 27.$$

Example 2. Find the volume bounded by the coördinate planes and the plane

$$x + y + z = 1.$$

The section in the plane x = constant is a triangle (Figure 250) extending from $y = 0$ to $y = 1 - x$. The area of this triangle is

$$A(x) = \int_0^{1-x} z\, dy = \int_0^{1-x} (1 - x - y)\, dy.$$

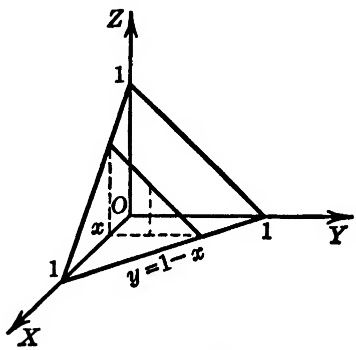

FIGURE 250.

The volume is formed by sections of this kind extending from $x = 0$ to $x = 1$. Thus

$$V = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$$
$$= \int_0^1 [y - xy - \tfrac{1}{2}y^2]_0^{1-x}\, dx$$
$$= \int_0^1 \tfrac{1}{2}(1 - x)^2\, dx$$
$$= [-\tfrac{1}{6}(1 - x)^3]_0^1$$
$$= \tfrac{1}{6}.$$

Example 3. Determine the region in the xy-plane over which the integral

$$\int_0^1 \int_{x^2}^{x} f(x, y)\, dy\, dx$$

extends.

In the first integration y varies from the curve $y = x^2$ to the line $y = x$. In the second integration x varies from 0 to 1. The region is that bounded by $y = x$ and $y = x^2$ (Figure 251).

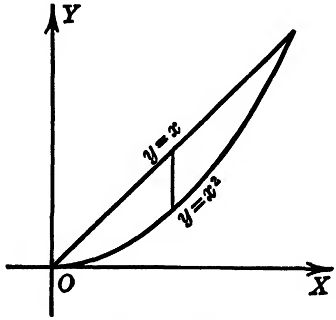

FIGURE 251.

Example 4. The integral

$$\int_{-2}^{2}\int_{-1}^{\frac{1}{8}x^3} f(x, y)\, dy\, dx$$

is extended over a certain region in the xy-plane. Determine an equivalent integral with the order of integration reversed.

The integral extends over the region (Figure 252) bounded by the line $y = -1$, the curve $y = \frac{1}{8}x^3$, and the line $x = 2$. With order of integration

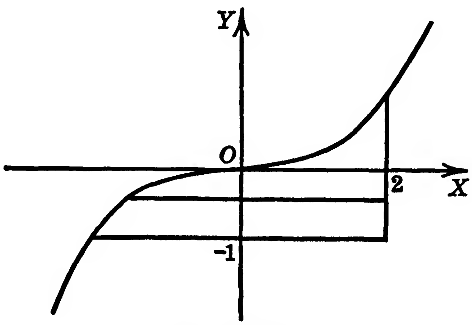

FIGURE 252.

changed, x varies from the curve $x = 2y^{\frac{1}{3}}$ to the line $x = 2$, and then y varies from -1 to 1. Thus

$$\int_{-1}^{1}\int_{2y^{\frac{1}{3}}}^{2} f(x, y)\, dx\, dy$$

is the expression required. That the two integrals are actually equivalent will follow from our later discussions (§204).

203. Multiple Integration. We now define a second method of integrating, or summing, over a region, and in the following section we will show that this is equivalent to repeated integration.

First consider a region A in the xy-plane and a function $f(x, y)$ defined at points of this region. Divide the plane into rectangles by lines parallel to the coördinate axes, the distances between

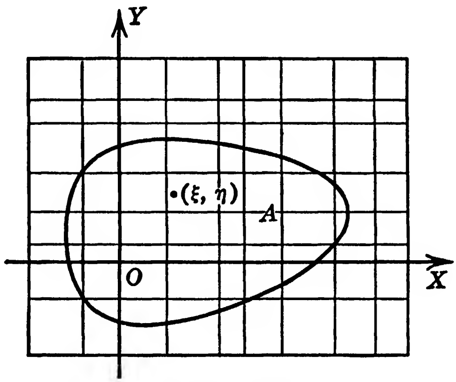

FIGURE 253.

consecutive parallel lines being equal or unequal. In each rectangle which lies entirely within A choose a point (ξ, η) and form the product

$$f(\xi, \eta)\,\Delta A,$$

where ΔA is the area of the rectangle. In case of a border rectangle, containing points both inside and outside A, a product of this kind may be included or omitted. Let

$$\Sigma f(\xi, \eta)\,\Delta A \tag{1}$$

be the sum of the products thus formed. When the number of divisions is increased in such a way that the edges of all the rectangles tend to zero, if this sum approaches a limit independent of the choice of dividing lines and of points (ξ, η) in the rectangles, and independent of which boundary rectangles are included and which omitted, the function $f(x, y)$ is said to be integrable in the region A and the limit is called the *double integral* of $f(x, y)$ over that region. It is indicated by the notation

$$\int_A f(x, y)\,dA. \tag{2}$$

In particular, if $f(x, y) = 1$, this integral defines the area of the region.

Similarly, being given a function $f(x, y, z)$ and a region V in space, we divide the region into rectangular elements of volume by planes parallel to the coördinate planes, choose a point (ξ, η, ζ) in each volume element Δv, and form the sum

$$\Sigma f(\xi, \eta, \zeta)\, \Delta v, \tag{3}$$

which may or may not include boundary elements. When the number of divisions increases in such a way that all the edges of the volume elements tend to zero, if this sum approaches a limit independent of the choice of dividing planes and points (ξ, η, ζ), and independent of which boundary elements are included and which omitted, the function $f(x, y, z)$ is said to be *integrable* in the region V and the limit is called the *triple integral* of $f(x, y, z)$ over that region. It is indicated by the notation

$$\int_V f(x, y, z)\, dv. \tag{4}$$

In particular, if $f(x, y, z) = 1$, this integral defines the volume of the region.

The sum (1) or (3) is not essentially different from the corresponding sum which defines the definite integral of a function of one variable. The principal difference is the more complicated nature of the boundary. In the integrals we shall consider we assume that the sum of the boundary elements of area or volume approaches zero as the process of subdivision proceeds.

204. Evaluation of a Multiple Integral by Repeated Integration. It remains to determine how a multiple integral is to be evaluated.

For brevity we use the notation

$$\int_A \int f(x, y)\, dy\, dx$$

to represent the repeated integral over a region A, the first integration being with respect to y, the second with respect to x.

If $f(x, y)$ is continuous in a finite region A,

$$\int_A f(x, y)\, dA = \int_A \int f(x, y)\, dx\, dy = \int_A \int f(x, y)\, dy\, dx, \tag{1}$$

provided the boundary of A is such that the integrals are defined.

The multiple integral and the repeated integrals over a given region are thus equal provided that they are all defined.

In proving this it is convenient to use two auxiliary functions. First let $g(x, y)$ be the function equal to $f(x, y)$ at all points of the

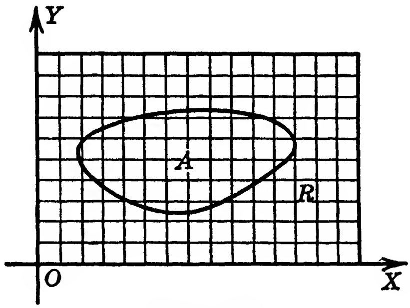

FIGURE 254.

region A and equal to zero at all other points. If R is some rectangle which contains A and has sides parallel to OX and OY, we then have

$$\int_R\int g(x, y)\,dy\,dx = \int_A\int f(x, y)\,dy\,dx. \tag{2}$$

By lines parallel to OX and OY divide R into rectangles so small that, if (x_1, y_1) and (x_2, y_2) are two points of A in the same rectangle,

$$|f(x_2, y_2) - f(x_1, y_1)| < \epsilon. \tag{3}$$

That this is possible for each positive value of ϵ is an extension of theorem 5, §14, to continuous functions of two variables. Next let $G(x, y)$ be a function which is constant in each rectangle of R and equal to the value of $g(x, y)$ at some point (ξ, η) in that rectangle. We then have

$$\int_R\int G(x, y)\,dy\,dx = \sum_A f(\xi, \eta)\,\Delta A, \tag{4}$$

the summation being of the form (§203) used in defining the double integral over A. By subtracting (4) from (2) we obtain

$$\int_A\int f(x, y)\,dy\,dx - \sum_A f(\xi, \eta)\,\Delta A$$
$$= \int_R\int [g(x,y) - G(x, y)]\,dy\,dx. \tag{5}$$

The right side of this equation is equal to a sum of integrals over the rectangles of R. The difference

$$|\, g(x, y) - G(x, y) \,|$$

is less than ϵ in each rectangle interior to A and in the boundary rectangles cannot exceed $2M$, where M is the maximum value of $f(x, y)$ in A. In rectangles outside A this difference is zero. The right side of (5) is therefore less than

$$\epsilon a + 2Mb,$$

where a is the area of the region R and b is the sum of the areas of the rectangles that contain boundary points of A. As the subdivisions diminish in size ϵ may be taken smaller and smaller, and b tends to zero. Equation (5) thus gives as limit

$$\int_A \int f(x, y)\, dy\, dx = \lim_{\Delta A \to 0} \sum_A f(\xi, \eta)\, \Delta A,$$

which is equivalent to

$$\int_A \int f(x, y)\, dy\, dx = \int_A f(x, y)\, dA.$$

In a similar way we obtain

$$\int_A \int f(x, y)\, dx\, dy = \int_A f(x, y)\, dA,$$

thus completing the proof.

205. Double Integrals, Rectangular Coördinates. Double integrals are used not only in plane summations but also in the calculation of space quantities which can be represented as double sums. Often it is possible to calculate the same quantity by one, two, or three integrations, the reduction in the number of integrations being accomplished by choosing sections such that part of the integrations can be performed mentally. Thus, if the section of a solid is a rectangle or a circle, it is unnecessary to determine its area by integration. In general it is advisable to use as few integrations as possible.

Example 1. Find the centroid of the plane area bounded by the parabolas $y^2 = 4 - x$, $y^2 = 4 - 2x$.

By symmetry the centroid is seen to be on the x-axis. Its abscissa is

$$\bar{x} = \frac{\int x\,dA}{\int dA},$$

the integrals being taken over the region between the parabolas. Expressing these as repeated integrals it is evident from the diagram (Figure 255) that the first integration should be with respect to x. For, if y varied first, the limits on the left of A would not be the same functions of x as those on the right. Hence

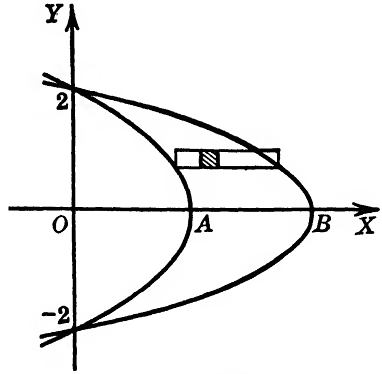

FIGURE 255.

$$\bar{x} = \frac{\int_{-2}^{2}\int_{\frac{1}{2}(4-y^2)}^{4-y^2} x\,dx\,dy}{\int_{-2}^{2}\int_{\frac{1}{2}(4-y^2)}^{4-y^2} dx\,dy} = \frac{\frac{64}{5}}{\frac{16}{3}} = \frac{12}{5}.$$

Example 2. Find the moment of inertia of the area bounded by the coördinate axes and the line $x + y = 2$ about the axis perpendicular to its plane at the origin.

The distance from the point (x, y) to the axis is $\sqrt{x^2 + y^2}$. Thus the moment of inertia is

$$\int (x^2 + y^2)\,dA$$

the integral being taken over the triangle (Figure 256) bounded by the axes and the line $x + y = 2$. Integrating first with respect to y we have

$$I = \int (x^2 + y^2)\,dA = \int_0^2\int_0^{2-x} (x^2 + y^2)\,dy\,dx = \tfrac{8}{3}.$$

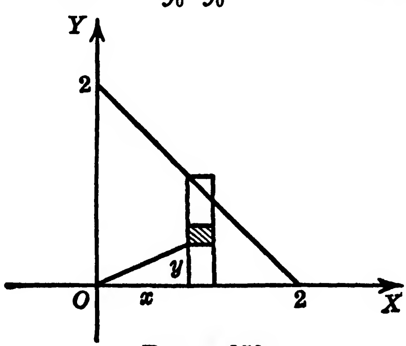

FIGURE 256.

Example 3. Find the volume bounded by the xy-plane and the paraboloid $z = 4 - y^2 - 4x^2$.

One-fourth of the required volume is shown in Figure 257. This is the limit of the sum of prisms of height z with bases $\Delta x \, \Delta y$ covering a quadrant of the ellipse $y^2 + 4x^2 = 4$. The volume is therefore

$$\int z\,dA = 4\int_0^1 \int_0^{2\sqrt{1-x^2}} (4 - y^2 - 4x^2)\,dy\,dx = 4\pi.$$

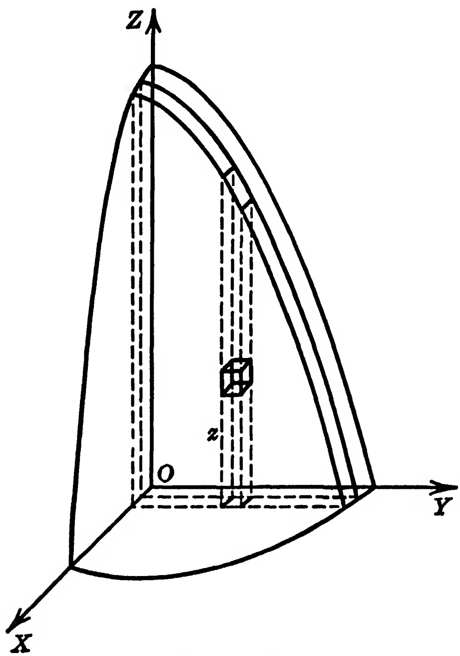

FIGURE 257.

206. Polar Coördinates. When polar coördinates are used the plane is cut into parts by the circles $r =$ constant and the lines $\theta =$ constant. The part bounded by two circles of radii r, $r + \Delta r$ and two lines θ, $\theta + \Delta\theta$ is approximately a rectangle of sides Δr, $r\,\Delta\theta$ and area

$$\Delta A = r\,\Delta\theta\,\Delta r,$$

the error in this approximation being of higher order than $\Delta\theta\,\Delta r$. If $f(r, \theta)$ is continuous, the sum

$$\Sigma f(r, \theta)\,\Delta A$$

taken for all elements ΔA in a region thus tends to the limit

$$\iint f(r, \theta) r\,dr\,d\theta$$

as Δr and $\Delta\theta$ tend to zero, the integral being taken over the region.

The limits in the first integration are the values of r at the ends A, B of the variable strip (Figure 258). The limits in the second

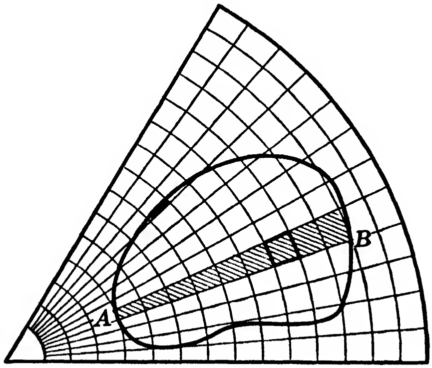

FIGURE 258.

integration are the values of θ determining the outside strips.

If it is more convenient the first integration may be with respect to θ. The integral then has the form

$$\iint f(r, \theta) r \, d\theta \, dr.$$

The first limits are now the values of θ at the ends of the strip

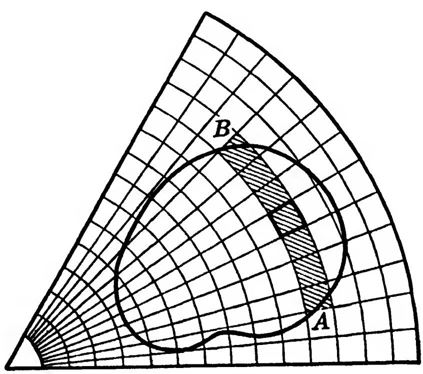

FIGURE 259.

determined by the circles of radii r, $r + \Delta r$ (Figure 259). The second limits are the extreme values of r.

Example 1. Find the value of the integral

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\, dy\, dx$$

by changing to polar coördinates.

The integral is taken over the semicircle $y = \sqrt{2ax - x^2}$ (Figure 260).

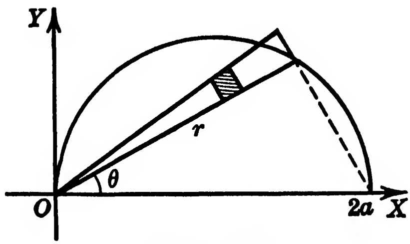

FIGURE 260.

In polar coördinates its equation is $r = 2a\cos\theta$. Also $x^2 + y^2 = r^2$. Hence

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^3\, dr\, d\theta = \tfrac{3}{4}\pi a^4.$$

Example 2. Find the centroid of the area outside the circle $r = a$ and inside the circle $r = 2a\cos\theta$.

By symmetry the centroid is on the x-axis. Its x-coördinate is

$$\bar{x} = \frac{\int x\, dA}{\int dA} = \frac{\int r\cos\theta\, dA}{\int dA},$$

the integration being taken over the area between the circles (Figure 261).

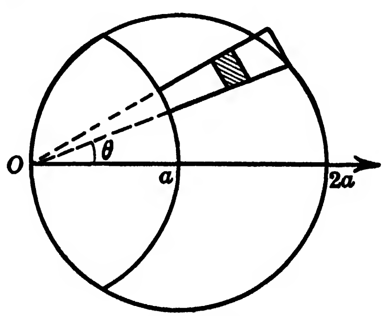

FIGURE 261.

By solving simultaneously we find that the circles intersect at $\theta = \pm\dfrac{\pi}{3}$.

Since the area is symmetrical with respect to the x-axis, we have

$$\bar{x} = \frac{2\int_0^{\frac{\pi}{3}}\int_a^{2a\cos\theta} r^2 \cos\theta \, dr \, d\theta}{2\int_0^{\frac{\pi}{3}}\int_a^{2a\cos\theta} r \, dr \, d\theta} = \frac{8\pi + 3\sqrt{3}}{4\pi + 6\sqrt{3}} a.$$

Example 3. Find the moment of inertia of the area of the cardioid $r = a(1 + \cos\theta)$ about the axis perpendicular to its plane at the origin.

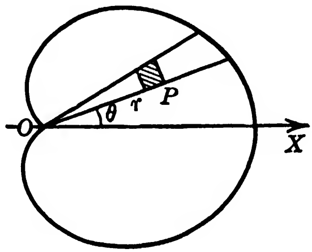

FIGURE 262.

The distance from any point $P(r, \theta)$ (Figure 262) to the axis of rotation is $OP = r$. Hence the moment of inertia is

$$I = \int r^2 \, dA = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3 \, dr \, d\theta = \tfrac{35}{16}\pi a^4.$$

Example 4. A homogeneous solid has the form generated by rotating the cardioid $r = a(1 + \cos\theta)$ about the x-axis. Find its moment of inertia about the x-axis.

When rotated about the x-axis the element of area

$$dA = r \, dr \, d\theta$$

generates a ring of volume

$$dV = 2\pi y \, dA$$

and moment of inertia

$$\rho y^2 \, dV = 2\pi\rho y^3 \, dA = 2\pi\rho r^3 \sin^3\theta \, dA,$$

where ρ is the density. The total moment of inertia is therefore

$$I = \int 2\pi\rho r^3 \sin^3\theta \, dA = 2\pi\rho\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^4 \sin^3\theta \, dr \, d\theta = \tfrac{64}{35}\pi\rho a^5.$$

207. Triple Integrals, Rectangular Coördinates. In a triple integral

$$\iiint f(x, y, z)\, dx\, dy\, dz$$

the integrations can be arranged in six different orders. The ease of performing these operations depends very much on the order in which they are taken. In a particular instance all orders should therefore be considered and the one chosen which involves the simplest integrations. Since the first two integrations sweep over the cross section determined by a constant value of the third variable, it is mainly a question of choosing these so that the resulting section is as simple as possible. By proper choice of sections most problems that lead to triple integrals can be solved by a single integration.

Example. Find the centroid of the solid bounded by the paraboloid $x^2 + 2z^2 = 4y$ and the plane $y = 2$.

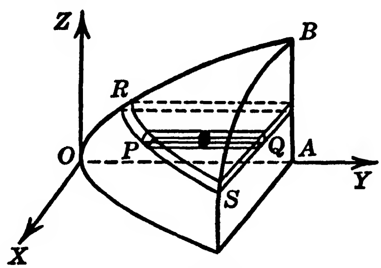

FIGURE 263.

One-fourth of the solid is shown in Figure 263. By symmetry $\bar{x}$ and $\bar{z}$ are zero. The y-coördinate of the centroid is

$$\bar{y} = \frac{\int y\, dV}{\int dV} = \frac{4\int_0^2 \int_0^{\sqrt{8-2z^2}} \int_{\frac{1}{4}(x^2+2z^2)}^2 y\, dy\, dx\, dz}{4\int_0^2 \int_0^{\sqrt{8-2z^2}} \int_{\frac{1}{4}(x^2+2z^2)}^2 dy\, dx\, dz} = \frac{4}{3}.$$

The limits for y are the values $y = \frac{1}{4}(x^2 + 2z^2)$ at P and $y = 2$ at Q. At S, $y = 2$ and $x = \sqrt{4y - 2z^2} = \sqrt{8 - 2z^2}$. The limits for x are therefore $x = 0$ at R and $x = \sqrt{8 - 2z^2}$ at S. The limits for z are $z = 0$ at A and $z = 2$ at B.

A simpler way to solve this problem is to note that the section $y =$ constant is an ellipse

$$\frac{x^2}{4y} + \frac{z^2}{2y} = 1$$

with semiaxes $a = \sqrt{4y}$, $b = \sqrt{2y}$, and area

$$A = \pi ab = 2\pi\sqrt{2}\,y.$$

The coördinate of the centroid is therefore

$$\bar{y} = \frac{\int_0^2 yA\,dy}{\int_0^2 A\,dy} = \frac{\int_0^2 y^2\,dy}{\int_0^2 y\,dy} = \frac{4}{3}.$$

208. Cylindrical Coördinates. If r, θ, z are cylindrical coördinates, the surfaces r = constant, θ = constant, z = constant divide space into elementary regions (PQ, Figure 264) which for

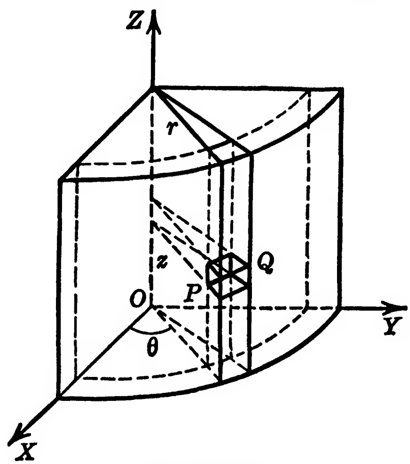

FIGURE 264.

small values of the increments are approximately rectangular parallelepipeds with edges Δr, $r\,\Delta\theta$, Δz and volume $\Delta v = r\,\Delta r\,\Delta\theta\,\Delta z$, the error in this approximation being an infinitesimal of higher order than $\Delta r\,\Delta\theta\,\Delta z$. When the increments all tend to zero, a sum of the form

$$\Sigma f(r, \theta, z)\,\Delta v$$

taken for all elements Δv in a region thus tends to the limit

$$\iiint f(r, \theta, z) r\,d\theta\,dr\,dz,$$

the integral being taken over the same region.

The integrations can be arranged in six different orders. The one should be chosen that involves the simplest integrations.

Example 1. A cone is rotated about a diameter of its base. Find the moment of inertia.

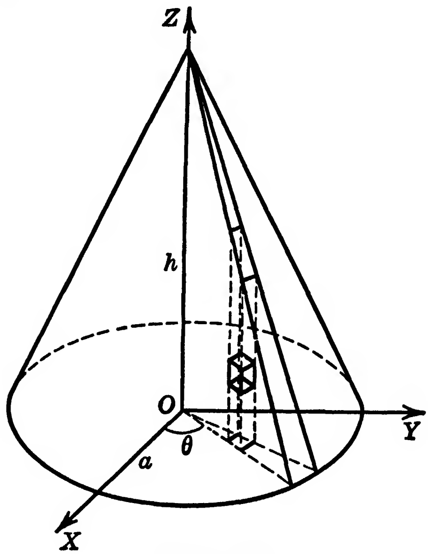

FIGURE 265.

Take the origin at the center of the base, let the z-axis be the axis of the cone, and take the moment of inertia about the x-axis. The equation of the conical surface is

$$z = \frac{h}{a}(a - r),$$

h being the altitude and a the radius of base. The distance from a point (x, y, z) to the x-axis is

$$\sqrt{y^2 + z^2} = \sqrt{r^2 \sin^2 \theta + z^2}.$$

The moment of inertia about the x-axis is therefore

$$I = \int_0^{2\pi} \int_0^a \int_0^{\frac{h}{a}(a-r)} \rho(r^2 \sin^2 \theta + z^2) r\, dz\, dr\, d\theta.$$

The limits in the first integration are the ends $z = 0$, $z = \dfrac{h}{a}(a - r)$ of a prism standing on the base $r\, d\theta\, dr$. The limits in the second integration are the boundaries $r = 0$, $r = a$ of a wedge-shaped slice extending from the axis to the perimeter of the base. The third integration sums these slices around the axis from $\theta = 0$ to $\theta = 2\pi$. By evaluating the integrals we find

$$I = \frac{\pi\rho a^2 h}{60}(3a^2 + 2h^2) = \frac{1}{20} M(3a^2 + 2h^2),$$

where $M = \frac{1}{3}\pi\rho a^2 h$ is the total mass of the cone.

Example 2. A hole of radius b is bored through a sphere of radius a, the axis of the hole being a diameter of the sphere. Find the moment of inertia of the ring thus formed about the axis of the hole.

In Figure 266 is shown one-eighth of the volume. The equation of the spherical surface is

$$r^2 + z^2 = a^2.$$

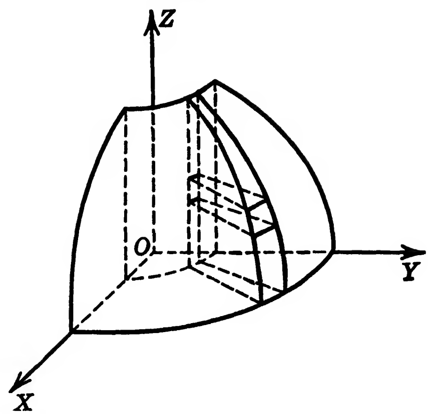

FIGURE 266.

The moment of inertia is

$$I = 2\int_0^{2\pi}\int_0^{\sqrt{a^2-b^2}}\int_b^{\sqrt{a^2-z^2}} \rho r^2\cdot r\,dr\,dz\,d\theta$$

$$= \frac{\pi\rho}{15}(8a^4 + 4a^2b^2 - 12b^4)\sqrt{a^2 - b^2},$$

ρ being the density.

Example 3. Find the volume bounded by the coördinate planes, the surface $z = x^2 + y^2$, and the plane $x + y = 1$.

In cylindrical coördinates the equation of the surface is $z = r^2$, and

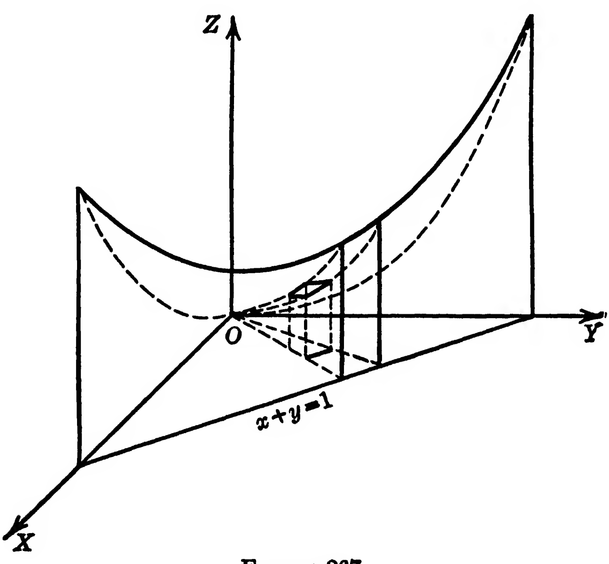

FIGURE 267.

that of the plane is $r = \frac{1}{2}\sqrt{2}\sec\left(\theta - \frac{\pi}{4}\right)$. The volume is therefore

$$V = \int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{2}\sqrt{2}\sec\left(\theta-\frac{\pi}{4}\right)}\int_0^{r^2} r\,dz\,dr\,d\theta = \tfrac{1}{6}.$$

209. Spherical Coördinates. In spherical coördinates the locus r = constant is a sphere with center at the origin, ϕ = constant is a cone with axis OZ, and θ = constant is a plane through the z-axis making the angle θ with OX.

In Figure 268, $PQRS$ is the region bounded by the spheres r, $r + \Delta r$, cones ϕ, $\phi + \Delta\phi$, and planes θ, $\theta + \Delta\theta$. When Δr, $\Delta\phi$, $\Delta\theta$

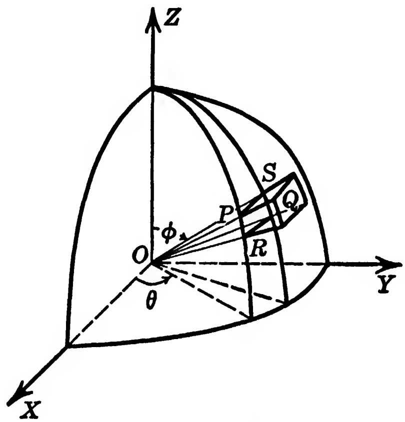

FIGURE 268.

are all very small this is approximately a rectangular parallelepiped with edges $PQ = \Delta r$, $PR = r\Delta\phi$, $PS = r\sin\phi\Delta\theta$. The volume of this region is approximately

$$PQ\cdot PR\cdot PS = r^2\sin\phi\;\Delta r\;\Delta\phi\;\Delta\theta,$$

the error in this approximation being an infinitesimal of higher order than $\Delta r\;\Delta\phi\;\Delta\theta$. When all the increments of the coördinates tend to zero, a sum of the form

$$\Sigma f(r, \theta, \phi)\;\Delta v,$$

taken for all volume elements in a given region, thus tends to the limit

$$\iiint f(r, \phi, \theta) r^2 \sin\phi\; dr\; d\phi\; d\theta,$$

the integral being taken over the same region.

Example. The inner and outer radii of a hemispherical shell of constant density are $r = a$, $r = b$. Find its center of gravity.

Take the center of the sphere as origin, and let the xy-plane be the face of the hemisphere. By symmetry the center of gravity is on the z-axis. Also

$$z = r \cos \phi.$$

Hence

$$\bar{z} = \frac{\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_a^b r \cos \phi \, r^2 \sin \phi \, dr \, d\phi \, d\theta}{\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_a^b r^2 \sin \phi \, dr \, d\phi \, d\theta} = \frac{3}{8}\frac{b^4 - a^4}{b^3 - a^3}.$$

210. Mean Values. In §133 we defined the mean value of $f(x)$ with respect to x between $x = a$ and $x = b$ as the quotient

$$\frac{\displaystyle\int_a^b f(x)\, dx}{\displaystyle\int_a^b dx}. \tag{1}$$

Similarly,

$$\frac{\displaystyle\int_A f(x, y)\, dA}{\displaystyle\int dA} \tag{2}$$

is called the mean value of $f(x, y)$ with respect to area in the region A of the xy-plane. If we sprinkle points over the region A, making the density (number per unit area) constant, take the average of the values of $f(x, y)$ at those points, and determine the limit as the number of points becomes infinite, the result will be the value determined by (2).

Similarly,

$$\frac{\displaystyle\int_V f(x, y, z)\, dV}{\displaystyle\int_V dV} \tag{3}$$

is the mean value of $f(x, y, z)$ with respect to volume in the region V of space.

The mean values determined by (2) and (3) are the mean with respect to area and mean with respect to volume. Other types of means are sometimes used, but, unless the contrary is explicitly stated, the term mean value is understood to signify mean with respect to area in a two-dimensional distribution and mean with respect to volume in a three-dimensional one.

Example. If p is the distance from a fixed point Q outside a sphere to a variable point P inside, find the average value of $\frac{1}{p}$ for all positions of P within the sphere.

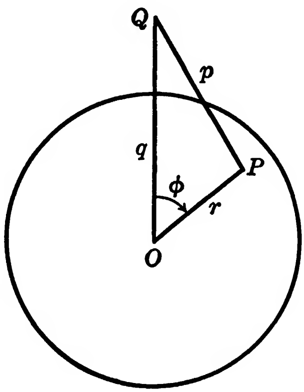

FIGURE 269.

Take the origin at the center of the sphere, the z-axis through Q, let a be the radius of the sphere and $q = OQ$. Using spherical coördinates,

$$p = \sqrt{q^2 + r^2 - 2qr\cos\phi}.$$

The average value of $\frac{1}{p}$ is

$$\frac{\int \frac{1}{p}\,dV}{\int dV} = \frac{\displaystyle\int_0^{2\pi}\int_0^a\int_0^\pi \frac{r^2\sin\phi\,d\phi\,dr\,d\theta}{\sqrt{q^2 + r^2 - 2qr\cos\phi}}}{\displaystyle\int_0^{2\pi}\int_0^a\int_0^\pi r^2\sin\phi\,d\phi\,dr\,d\theta} = \frac{1}{q}.$$

The average value of the function $\frac{1}{p}$ is thus the value $\frac{1}{q}$ which this function takes at the center of the sphere.

211. Area of a Surface. On the surface

$$z = f(x, y)$$

let S be a region at all points of which $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are continuous functions of x and y.

Let P be a point of S, and Q its projection on the xy-plane. When Q describes an elementary area ΔA in the xy-plane, P describes an area ΔS on the surface. When ΔA is sufficiently small,

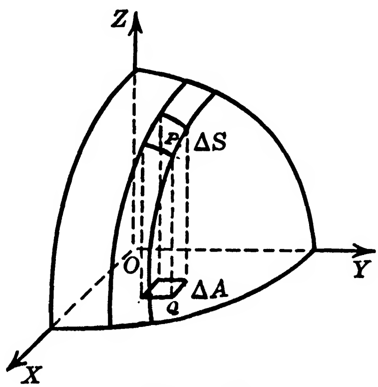

FIGURE 270.

ΔS lies approximately in the tangent plane at P. The areas ΔA and ΔS thus satisfy approximately the equation

$$\Delta A = \Delta S \cos \gamma,$$

where γ is the angle between the xy-plane and the tangent plane at P, that is, the angle between the z-axis and the normal at P.

The total area of the region on the surface is approximately

$$\sum \Delta S = \sum \frac{\Delta A}{\cos \gamma}.$$

As the elementary areas tend to zero, these approximations become more and more accurate. Hence in the limit we have

$$S = \int_A \frac{dA}{\cos \gamma}$$

as the area of surface, the integral being taken over the projection on the xy-plane. To evaluate this integral we choose appropriate

coördinates and determine $\cos \gamma$ from the equation of the surface. In a similar way we could obtain a formula for the area of surface in terms of its projection on any other plane. The plane of projection may to a certain extent be chosen arbitrarily. It should, however, be such that $\cos \gamma$ does not become zero (at least does not change sign) at points of S.

Example. Find the surface of the sphere $x^2 + y^2 + z^2 = a^2$ inside the cylinder $x^2 + y^2 = ax$.

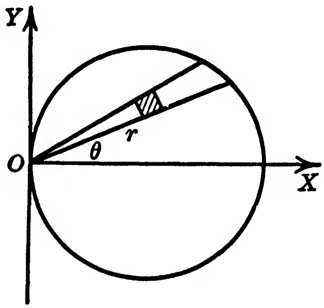

FIGURE 271.

The projection on the xy-plane is the circle $x^2 + y^2 = ax$, or in polar coördinates, $r = a \cos \theta$ (Figure 271). From the equation of the sphere we have

$$\cos \gamma = \frac{z}{a} = \frac{\sqrt{a^2 - x^2 - y^2}}{a} = \frac{\sqrt{a^2 - r^2}}{a}.$$

The area required is therefore

$$S = 4\int_0^{\frac{\pi}{2}}\int_0^{a \cos} \frac{ar\,dr\,d\theta}{\sqrt{a^2 - r^2}} = (2\pi - 4)a^2.$$

The factor 4 is introduced because the cylinder intersects the sphere below as well as above the xy-plane, and the integral represents only half the area above the plane.

PROBLEMS

Find the values of the following integrals:

1. $\displaystyle\int_1^3\int_0^{y-1} (y - 2x)\,dx\,dy.$

2. $\displaystyle\int_1^2\int_x^{2x} \frac{dy\,dx}{(x + y)^2}.$

3. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{a \cos \theta} \sin \theta\,dr\,d\theta.$

4. $\displaystyle\int_0^2\int_0^z\int_0^{y+z} dx\,dy\,dz.$

5. Determine the volume bounded by the coördinate planes and the plane

$$x + 2y + 3z = 6.$$

6. A solid is bounded above by the surface

$$z = x - y^2$$

and below by the xy-plane. By integration determine the area $A(x)$ of the section cut from the solid by the plane x = constant. By a second integration determine the volume of the solid between the planes $x = 0$ and $x = 4$.

In each of the following problems determine the region over which the integral extends:

7. $\int_0^1 \int_0^y f(x, y)\, dx\, dy.$

8. $\int_0^{\frac{\pi}{3}} \int_1^{2\cos x} f(x, y)\, dy\, dx.$

9. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} f(r, \theta)\, dr\, d\theta.$

10. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} dz\, dy\, dx.$

In each of the following problems determine an equivalent integral with the order of integration reversed:

11. $\int_0^2 \int_{2x}^4 f(x, y)\, dy\, dx.$

12. $\int_0^1 \int_0^{\sqrt{2-2y^2}} f(x, y)\, dx\, dy.$

13. $\int_1^4 \int_2^{2\sqrt{x}} f(x, y)\, dy\, dx.$

14. $\int_0^2 \int_{y^2}^{2y} f(x, y)\, dx\, dy.$

15. Find the centroid of the area bounded by the parabolas $y^2 = x + 3$, $y^2 = 5 - x$.

16. Find the centroid of the area bounded by the parabola $y^2 = 2x + 1$ and the line $y = 1 - x$.

17. An area is bounded by the parabola $y^2 = 4ax$ and the line $x = a$. Find the moment of inertia of this area about the axis perpendicular to its plane at the origin.

18. A thin plate of mass M has the form of a right triangle of sides a and b. Find its moment of inertia about the axis perpendicular to its plane at the vertex of the right angle.

19. Find the moment of inertia of a cube of side a and mass M about an edge.

20. Find the volume of the pyramid bounded by the coördinate planes, and the plane $x + 2y - 3z = 6$.

21. Find the volume under the paraboloid $z = x^2 + y^2$ and over the square bounded by the lines $x = \pm 1$, $y = \pm 1$ in the xy-plane.

22. Find the volume under the plane $2x + 2y + z = 8$ and over the triangle bounded by the lines $x = 0$, $y = 2$, $x = y$ in the xy-plane.

23. A solid is bounded by the planes $y = 0$, $z = 0$, $x = z$, $x + y = 1$. Find its volume.

24. A solid is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 2$, and the surface $z = (y - 2)^2$. Find its volume.

25. Find the volume above the xy-plane bounded by the surface $z = x^2 - y^2$ and the planes $x = 1$, $x = 3$.

26. Find the volume bounded by the planes $y = 0$, $y = x$ and the cylinder $z^2 = a^2 - ax$.

Find the values of the following integrals by changing to polar coördinates:

27. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2)\, dx\, dy.$

28. $\displaystyle\int_0^a \int_0^{\sqrt{ax-x^2}} dy\; dx.$

29. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx.$

30. $\displaystyle\int_0^a \int_0^x \sqrt{x^2 + y^2}\, dy\, dx.$

31. $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$

32. $\displaystyle\int_0^1 \int_{x^2}^x \frac{dy\, dx}{\sqrt{x^2 + y^2}}.$

33. Find the area inside a loop of the curve $r = a \cos 2\theta$ and outside the circle $r = \dfrac{a}{2}$.

34. Find the area cut from the lemniscate $r^2 = 2a^2 \cos 2\theta$ by the line $x = \dfrac{\sqrt{3}}{2} a$.

35. Find the centroid of the area within the curve $r = 2 \cos \theta + 3$.

36. Find the centroid of the area bounded by the circle $r = a \cos \theta$, the x-axis, and the line $y = x$.

37. Find the centroid of the area bounded by the x-axis and the circles $r = a \cos \theta$, $r = a \sin \theta$.

38. Find the moment of inertia of a circular plate of radius a and mass M about an axis perpendicular to its plane at a point of its circumference.

39. Find the moment of inertia of the area within one loop of the curve $r^2 = a^2 \sin \theta$ about the axis perpendicular to its plane at the origin.

40. A circular sector has the radius a and central angle 2α. Find the moment of inertia of its area about the bisector of the angle.

41. Find the moment of inertia of the area within one loop of the curve $r^2 = 2a^2 \cos 2\theta$ about the axis perpendicular to its plane at the origin.

42. A circular sector of radius a and angle α is rotated about one of its bounding radii. Find the volume generated.

43. The area outside the circle $r = a$ and inside the circle $r = 2a \cos \theta$ is rotated about the initial line. Find the volume generated.

44. An anchor ring is generated by rotating a circle of radius a about an axis at distance b (greater than a) from the center. If its mass is M, find its moment of inertia about the axis.

45. A pyramid of constant density ρ is bounded by the coördinate planes and the plane $x + 2y + 3z = 6$. Find its moment of inertia about the z-axis.

46. Find the volume bounded by the cylinder

$$z^2 = 1 - x - y$$

and the coördinate planes in which x, y, z are positive.

47. Find the volume bounded by the cylinders $y^2 + z^2 = a^2$, $z^2 = a^2 - ax$, and the plane $x = 0$.

48. Find the volume bounded by $x = 0$, $z = 0$, $z = 1 - xy$, $y^2 = 1 - x$.

49. Find the centroid of an octant of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

50. Find the moment of inertia of the ellipsoid in the preceding problem about the z-axis.

51. A solid of constant density is bounded by the xy-plane and the paraboloid $x^2 + 2y^2 = 4 - z$. Find its moment of inertia about the z-axis.

52. A hole of radius a is bored through a sphere of radius $2a$. If the axis of the hole passes through the center of the sphere, find the volume cut out.

53. Find the moment of inertia of a sphere of radius a and mass M about a diameter.

54. A cone has altitude h, radius of base a, and mass M. Find its moment of inertia about an axis through the vertex parallel to the base.

55. Find the moment of inertia of a sphere of radius a and mass M about a tangent line.

56. A segment is cut from a sphere of radius a and density ρ by a plane at distance $\frac{1}{3}a$ from the center. Find its moment of inertia about a diameter parallel to this plane.

57. A spherical shell of inner radius a, outer radius b, and density ρ is cut by a plane at distance $c < a$ from the center. Find the moment of inertia of the segment cut off about the diameter perpendicular to this plane. Consider the solid as the difference of two spherical segments.

58. Find the volume bounded above by the sphere $r^2 + z^2 = 5$ and below by the paraboloid $r^2 = 4z$.

59. Find the volume bounded by the xy-plane, the cylinder $r = 2a \cos \theta$, and the cone $r = z$.

60. Find the volumes of the two parts into which the sphere $r^2 + z^2 = a^2$ is cut by the cone $r + z = a$.

61. Find the volumes of the parts into which the sphere $r^2 + z^2 = 4az$ is cut by the paraboloid $r^2 = az$.

62. Find the volume bounded by the xy-plane, the paraboloid $az = x^2 + y^2$, and the cylinder $x^2 + y^2 = a^2$.

63. Find the volume of the cylinder $x^2 + y^2 = 2ax$ intercepted between the xy-plane and the paraboloid $x^2 + y^2 = 2az$.

64. Find the volume bounded by the cylinder $r = a \cos \theta$ and the planes $\theta = 0$, $\theta = \frac{\pi}{2}$, $z = 0$, $z = x + y$.

65. Find the volume bounded by the surface $z = e^{-(x^2+y^2)}$ and the xy-plane.

66. Find the volume cut from the paraboloid $z = x^2 + y^2$ by the plane $z = 2x$.

67. Find the moment of inertia of a cylinder of radius a and mass M about its axis if its density is proportional to the distance from its axis.

68. Find the center of gravity of a cone of altitude h if its density is proportional to the distance from its base.

69. Find the center of gravity of a cone of radius a and altitude h if its density is proportional to the distance from its axis.

70. Find the centroid of the volume cut from a sphere of radius a by two planes through the center including an angle of 60°. Use spherical coördinates, and take the planes as $\theta = \pm \frac{\pi}{6}$.

71. Find the centroid of the region within a sphere of radius a and a cone of vertical angle 2α, if the vertex of the cone is the center of the sphere.

72. The vertex of a cone of vertical angle 60° is on the surface of a sphere of radius a, and its axis is a diameter of the sphere. Find the moment of inertia of the volume common to the cone and sphere about the axis of the cone.

73. In spherical coördinates the equation $r = a \sin 2\theta$ represents a surface. Find the volume between $\theta = 0$ and $\theta = \frac{\pi}{2}$ bounded by this surface.

74. Find the volume enclosed by the surface $r = a \sin \phi$.

75. From a spherical shell of inner radius a and outer radius b a segment is cut by a plane at distance c (less than a) from the center. Find the volume of the segment. Use spherical coördinates, and integrate first with respect to ϕ.

76. Find the center of gravity of a hemisphere of radius a if its density is proportional to the distance from its center.

77. Find the center of gravity of a right circular cone of altitude h and radius of base a if its density is proportional to the distance from the vertex.

78. Find the mass of a cube of side $2a$ if its density is k times the square of the distance from its center.

79. Find the center of gravity of a cube of side a if its density is proportional to the square of the distance from one corner of the base. Take the edges through this corner as axes.

80. The density of a sphere of radius a and mass M is proportional to the distance from a diametral plane. Find its moment of inertia about the diameter perpendicular to that plane.

81. Find the volume above the xy-plane bounded by the sphere $r = a$, the cone $\phi = \theta$, and the plane $\theta = 0$.

82. Let p be the distance from a variable point P in a circle of radius a to a fixed point Q at distance q from the center of the circle. Find the mean value of p^2 for all positions of P within the circle.

83. Find the mean distance from one corner of a square of side a to points inside the square.

84. Find the mean square of the distance from the center of a cube of side a to points inside the cube.

85. Find the mean distance from a point on the surface of a sphere of radius a to points inside the sphere.

86. On a circle of radius a a large number of chords are measured. Assuming the ends of the chords to be uniformly distributed around the circumference, find the average length of the chords.

87. Find the area of the triangle cut from the plane

$$x + 2y + 3z = 6$$

by the coördinate planes.

88. Find the area cut from the plane $x + y + z = a$ by the cylinder $x^2 + y^2 = a^2$.

89. Find the area above the xy-plane cut from the cone $x^2 + y^2 = z^2$ by the cylinder $x^2 + y^2 = 2ax$.

90. Find the area of the surface of the paraboloid $y^2 + z^2 = 2ax$ cut off by the plane $x = a$.

91. Find the area of the surface $z^2 = 2xy$ above the xy-plane bounded by $y = 1$ and $x = 2$.

92. Find the area on the conical surface $x^2 + y^2 = z^2$ between the xy-plane and the plane $x + 2z = 3$.

93. Find the area cut from the plane by the cone in the preceding problem.

94. Find the area above the xy-plane cut from the cone $x^2 + y^2 = z^2$ by the sphere $x^2 + y^2 + z^2 = 2ax$.

95. Find the mean distance from points on the surface of a sphere to a fixed point on the surface.

96. If p is the distance from a fixed point Q inside a sphere of radius a to a variable point P on the surface, find the mean value of $\dfrac{1}{p}$ for all positions of P on the surface.

ANSWERS TO PROBLEMS

Chapter I. Pages 21–26

4. $A = \dfrac{C^2}{4\pi}$.

5. $V = \dfrac{A^{\frac{3}{2}}}{6\sqrt{\pi}}$.

8. $|x| < 2$.

9. $-2 < x < 4$.

10. $x = \pm\sqrt{\dfrac{y}{1 - y}}$, $0 \leqq y < 1$.

11. $y = 1 \pm \sqrt{1 - x}$, $x \leqq 1$.

12. $f(0) = 0, f(1) = 0, f(2) = 2, f(-2) = 6$.
13. $f(0) = 1, f(1) = 2, f(2) = 4, f(-2) = \frac{1}{4}$.
14. 0.
15. $f(x) + 2 = 4 - x, f(x + 2) = -x$.
16. $f\left(\dfrac{1}{x}\right) = \dfrac{1 - 2x}{1 + x}, \dfrac{1}{f(x)} = \dfrac{x + 1}{x - 2}$.
17. $f(x^2) = x^2 + 1, [f(x)]^2 = x^2 + 2x + 1$.

28. 1.
29. 0.
30. 1.
31. 0.
32. $\frac{10}{9}$.
33. $3\frac{13}{99}$.
34. $2\frac{1}{99}$.
35. 1.
36. 0.
37. -1.
38. $\frac{1}{2}$.
39. 1.
40. 0.
41. 0.
42. 1.
43. $\dfrac{1}{1 - x}$.
44. $\frac{1}{2}$.
45. $\dfrac{p^2}{4\pi}$.
46. $\lim S_n = 3l, \lim A_n = 0$.
47. $2a$.
48. l.
49. Yes, no.
51. 2.
52. $5h$.
53. $5 + 2\sqrt{6} = 9.899$ sec.
54. 1.
55. 7.
56. 3.
57. -6.
58. $\frac{3}{2}$.
59. 0.
60. 1.
61. 1.
62. 0.
63. $\frac{1}{4}$.
64. $\dfrac{1}{2\sqrt{a}}$.
65. 1.
66. 0.
67. 0.
68. No limit.
69. 3.
70. 1.
71. 0.
72. 1.
73. 1.
74. 2.
75. $\frac{3}{4}$.
76. 0.
77. 1.
78. No limit.
79. $\frac{1}{2}$.
80. $-\frac{1}{2}$.
81. 0.
82. 0.
83. $\sqrt{2}$.
84. 0
85. 2.
86. 0.

Chapter I (Continued)

87. 0. **88.** 0. **89.** 1.
90. 1. **91.** 0, -1. **92.** -1.
93. 0. **94.** None. **95.** $n\pi \pm \frac{\pi}{2}$.
96. $n\pi \pm \frac{\pi}{4}$. **97.** $x \leqq 0$.

Chapter II. Pages 64–72

1. $10 + 16t$, 10. **2.** 34.4 ft./sec., 36 ft./sec.
3. 13 ft./sec., 12 ft./sec. **4.** 16.
5. $(50 + 16t)$ ft./sec., $(50 + 32t)$ ft./sec.
6. $t = \frac{b}{k}$, av. speed $= \frac{b}{2}$. **7.** 2.
8. -3. **9.** $45°$. **10.** $135°$.
11. 3. **12.** 4. **13.** 3.
14. $3x_1^2$. **15.** (a) 0, (b) 1. **16.** 1.
17. $b - a$. **18.** 7. **19.** -7.
26. $(3x^2 - 1)\,\Delta x$. **27.** $(2x + 3)\,\Delta x$. **28.** $\frac{\Delta x}{2\sqrt{x}}$.
29. $-\frac{\Delta x}{(x - 1)^2}$. **32.** (a) $ds = 60$. (b) $\Delta s = 60$.
33. (a) 10,000. (b) 10,000. **34.** $dV = 4\pi r^2 t$.
35. $6x^2 t$, $\Delta x = 2t$, $dV = 6x^2 t$. **36.** $\frac{1}{12}A$ ft.^{3}
37. 2π in.^{2}
38. (a) $\Delta f(x) = 0.09902$,
$df(x) = 0.10000$,
$\Delta f(x) - df(x) = -0.00098\,\Delta x$.
(b) $\Delta f(x) = 0.0009999$,
$df(x) = 0.0010000$,
$\Delta f(x) - df(x) = -0.00001\,\Delta x$.
39. (a) $0.00909 = 0.0909\,\Delta x$.
(b) $0.000000999 = 0.000999\,\Delta x$.
40. $4x^3 - 12x^2 + 4x + 4$. **41.** $x^3 - 2x^2 + x$.
42. $12(x^2 - 2x + 1)$. **43.** $x^2 + 2x$. **44.** $\frac{3x + 1}{2\sqrt{x}}$.
45. $\frac{2x - 1}{\sqrt{8x^3}}$. **46.** $\frac{3(x^2 - 1)}{\sqrt{x^3}}$. **47.** $6(2x - 1)^2$.
48. $4(2x + 1)(x^2 + x + 1)^3$. **49.** $\frac{x - 1}{\sqrt{x^2 - 2x + 3}}$.
50. $6(2x - 1)(x - 1)^2$. **51.** $3x^2$.
52. $(8x - 2)(x + 1)(2x - 3)$. **53.** $\frac{3x}{2\sqrt{x + 1}}$.
54. $\frac{2(1 - x^2)}{\sqrt{2 - x^2}}$. **55.** $15x^3\sqrt{x^2 + a^2}$. **56.** $\frac{1}{(2x + 3)^2}$.

Chapter II (Continued)

57. $\dfrac{2x}{(1 - x^2)^2}$.
58. $\dfrac{1}{x^2\sqrt{4x^2 - 1}}$.
59. $\dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.
60. $4x^2(x^3 - 2)^{\frac{1}{3}}$.
61. $\dfrac{1}{(x^2 + 2x + 2)^{\frac{3}{2}}}$.
62. $\dfrac{1}{x^4\sqrt{x^2 + 1}}$.
63. $\dfrac{2x}{(2x^6 + 1)^{\frac{4}{3}}}$.
64. $\dfrac{4(3x - 1)^3}{x^5}$
65. $12 \cos 4x$.
66. $-2 \sin\left(x + \dfrac{\pi}{3}\right)$.
67. $1 - \cos x$.
68. $x \cos x$.
69. $4 \sin (2x + 3) \cos (2x + 3)$.
72. $45°$.
73. $x = 1, -\frac{1}{2}, -2$.
75. Yes.
76. No.
77. 0.05, 5%.
78. 0.1%.
79. $\dfrac{l}{43{,}200}$.
80. 3%.
81. $\frac{1}{2}$%.
82. $-(2x + 3)^{-\frac{3}{2}}$.
83. $-4 \sin 2x$.
84. $10(2x^2 - 1)(2 - x^2)^{\frac{1}{2}}$.
85. $\dfrac{2(y - 1)}{(x + 3)^2}$.
86. $-\dfrac{a^2}{y^3}$.
87. $-\dfrac{5}{y^3}$.
88. $\dfrac{1}{y^3}$.
89. $-\dfrac{30}{(x + 2y)^3}$.
90. 0.
91. $\dfrac{1}{2x^{\frac{3}{2}}}$.
92. $-\csc^3 \theta$.
93. $-\dfrac{4t^3}{(t^2 - 1)^3}$.
94. $t^2 \dfrac{d^3x}{dt^3}$.

96. Increasing if $x > 2$.
Decreasing if $x < 2$.
Concave upward for all x.

97. Increasing if $|x| > 1$.
Decreasing if $|x| < 1$.
Concave upward if $x > 0$.
Concave downward if $x < 0$.

98. Increasing if $x < -2$ or $x > 0$.
Decreasing if $-2 < x < 0$.
Concave upward if $x > -1$.
Concave downward if $x < -1$.

99. Increasing for all x.
Concave upward if $x < 0$.
Concave downward if $x > 0$.

100. Increasing if $x > 3$.
Decreasing if $x < 3$.
Concave upward if $x < 0$ or $x > 2$.
Concave downward if $0 < x < 2$.

101. Increasing if $x > -1$.
Decreasing if $x < -1$.
Concave upward for all x.

104. $A = 100$ in.^{2}
105. $P = 40$ in.
106. Side of base 10 in., depth 5 in.
107. Side of base 4 in., depth 1 in.
108. $h = r$.
109. $V = 8$ in.^{3}
110. $h = \frac{2}{3}\sqrt{3}$.
111. Arc is twice radius.
112. $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$.
113. 8 in.
114. Width is twice depth.
115. $\sqrt{6}$ in.
116. $r = 2h$.

Chapter II (Continued)

117. Ratio of length to width $= a/b$.
118. One-third the way from the lesser to the greater source.

119. Four hours later.
120. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.
121. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.
124. Speed through water 7.5 mi./hr
125. Speed through water is zero.
126. All circle.
127. (1, 0).
128. (1, 0).
129. $y = 1$.
130. $y = 1$.
131. $V = 48\pi$.
132. -48.

133. (*a*) $1 < t < 3$. (*b*) $t < 2$.
134. (*a*) toward, (*b*) toward, (*c*) increasing.
135. (*a*) away, (*b*) toward, (*c*) decreasing.
136. (*a*) away, (*b*) toward, (*c*) decreasing.
137. (*a*) away, (*b*) away, (*c*) increasing.
138. (*a*) toward, (*b*) away, (*c*) decreasing.
139. (*a*) away, (*b*) away, (*c*) increasing.
141. 32 ft./sec.^{2}
142. $\dfrac{R^2g}{r^2}$, where R is the radius of the earth.

145. $2\frac{1}{4}$ ft./sec.
146. $8\frac{1}{3}$ ft./sec., $3\frac{1}{3}$ ft./sec.
147. $\dfrac{64\sqrt{3}}{3}$ ft./sec.
148. 8 ft./sec.
149. 5 ft.
150. $\dfrac{5}{9\pi}$ in./sec.
151. $\dfrac{5\sqrt{5}}{3}$ in.^{2}/sec.
152. 4,356,000 ft.^{3}/day.
153. $\frac{5}{2}$.
154. 1.
155. 0.
156. 4.
157. 3.
158. 2.
161. $\frac{10}{9}$.
162. $\frac{5}{2}a^{\frac{3}{2}}$.
163. $\frac{1}{6}$.
164. 0.
165. $\frac{1}{4}$.
166. $-\frac{4}{3}$.
167. 0.
168. 0.
169. -1.
170. 0.

Chapter III. Pages 96–101

1. $x^3 - 2x^2 + 2x + C$.
2. $2x^4 + 2x^3 - x^2 + 5x + C$.
3. $\dfrac{y^3}{3} - \dfrac{1}{y} + C$.
4. $\frac{2}{3}(x + 3)\sqrt{x} + C$.
5. $\frac{1}{3}(2x)^{\frac{3}{2}} - (2x)^{\frac{1}{2}} + C$.
6. $\frac{1}{6}(2t + 1)^3 + C$.
7. $\dfrac{x^2}{2} - 2x + \dfrac{3}{x} + C$.
8. $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C$.
9. $\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + C$.
10. $-\dfrac{1}{u + 1} + C$.
11. $-\dfrac{1}{3(3x + 2)} + C$.
12. $\frac{1}{3}(2t + 3)^{\frac{3}{2}} + C$.
13. $\frac{2}{3}\sqrt{3x + 2} + C$.
14. $\sqrt{y^2 - 1} + C$.

Chapter III (Continued)

15. $-\dfrac{1}{6(3x^2+2)}+C.$ **16.** $\frac{1}{9}(3x^2+2)^{\frac{3}{2}}+C.$

17. $-\dfrac{1}{3(a^3+x^3)}+C.$ **18.** $\frac{1}{24}(4x^4+5)^{\frac{3}{2}}+C.$

19. $2\sqrt{x^2+x+1}+C.$ **20.** $-\dfrac{1}{3}\left(1+\dfrac{1}{t}\right)^3+C.$

21. $\frac{1}{2}\sin 2\theta+C.$ **22.** $-\frac{1}{2}\cos(2x+3)+C.$

23. $\frac{1}{2}\sin^2 t+C.$ **24.** $\frac{2}{3}(1-\cos y)^{\frac{3}{2}}+C.$

25. $30t+\frac{1}{2}gt^2.$ **26.** $h_0+v_0t-\frac{1}{2}gt^2.$ **27.** 138 ft.
28. $x=4.$ **29.** 18 ft./sec. **30.** 2 ft./sec.
31. 38.72. **32.** $y=x^2-3x+5.$ **33.** $y=2-\cos x.$

34. 4. **35.** $y=\frac{1}{6}x^3-\frac{1}{2}x^2+\frac{1}{2}x+\frac{13}{6}.$

36. $y=x^3+2x-1.$ **41.** 6. **42.** $\frac{4}{3}.$
43. $12\frac{2}{3}.$ **44.** 2. **45.** 0.
46. $\frac{1}{3}a^3.$ **47.** 2. **48.** 18.
49. $\frac{1}{2}.$ **50.** $\frac{32}{3}.$ **51.** $\frac{4}{3}.$
52. 15. **53.** $\frac{4}{3}.$ **54.** 2.
55. $\frac{32}{3}.$ **56.** $\frac{32}{3}.$ **57.** $\frac{4}{3}.$
58. $\frac{4}{3}.$ **59.** $4\frac{1}{2}.$ **60.** $\frac{8}{3}\pi.$
61. $\frac{512}{15}\pi.$ **62.** $2\pi a^3.$ **63.** $\frac{3}{5}\pi.$

65. $\frac{1}{3}\pi(2a^3-3a^2h+h^3),\ \frac{1}{3}\pi(2a^3+3a^2h-h^3).$ **66.** $\frac{4}{3}\pi ab^2.$

67. $\frac{16}{15}\pi.$ **68.** $\frac{56}{15}\pi.$ **69.** $\frac{32}{15}\pi a^3.$
70. $\frac{16}{5}\pi a^3.$ **71.** $\frac{112}{15}\pi a^3.$ **72.** $\frac{32}{3}\pi a^3.$
73. $9\pi.$ **74.** $\frac{1}{5}\pi(b^5-a^5).$ **75.** $\frac{16}{3}r^3.$
76. $2a^3\sqrt{3}.$ **77.** 6. **78.** $a^2h.$

79. $\frac{8}{3}a^3$, where a is the radius of the circles. **80.** $\frac{1}{2}\pi a^2h.$

81. $\frac{2}{3}a^3\tan\alpha$, where a is the radius of the cylinder.

82. $\frac{1}{3}a^3.$ **83.** 10 tons. **84.** 7.5 tons.
85. $\frac{1}{3}wbh^2.$ **86.** $1388\frac{8}{9}$ tons. **87.** 13.3 tons.
88. $\dfrac{k(b-a)^2}{2a}.$ **89.** $128\pi w.$ **90.** $9600\pi w.$
91. $\frac{1}{12}wAh^2.$ **92.** $(2.1)\ 10^8$ ft-lb. **93.** $(1.06)\ 10^8$ ft-lb.
94. 140,000 ft-lb. **95.** 463 ft-lb.

Chapter IV. Pages 137–145

1. $x=\frac{1}{2}(x_1+x_2),\ y=\frac{1}{2}(y_1+y_2).$ **2.** (2, 0).
3. $x=4,\ y=-3.$ **4.** $(0,\frac{1}{3}),\ (-4,7).$

5. (9, 14), (−3, −22).

6. $x=\dfrac{r_1x_2+r_2x_1}{r_1+r_2},\ y=\dfrac{r_1y_2+r_2y_1}{r_1+r_2}.$

7. $x=x_1+r(x_2-x_1),\ y=y_1+r(y_2-y_1).$

Chapter IV (Continued)

8. $x = 15, y = -3.$ **9.** $y - 4x + 11 = 0.$ **10.** $y = 2.$
11. $x = 3.$ **12.** $y - 3x + 7 = 0.$ **13.** $x = 2.$
18. $3x + 4y = 5.$ **19.** $5x + 4y = 22.$ **21.** $x - 3y - 7 = 0.$

22. $x + 3y - 7 = 0.$ **23.** $120°.$
25. $\tan A = \frac{1}{2}, \tan C = 2, B = 90°.$ **26.** 1, 2, 3.

29. $x - 3y - 6 = 0.$
31. $x - 3y + 17 = 0, 3x + y - 9 = 0.$
32. $x + 2y = 0, 2x - y = 0.$

33. Tangent: $y - 12x + 16 = 0.$
Normal: $x + 12y - 98 = 0.$
34. Tangent: $3x + 2y - 12 = 0.$
Normal: $2x - 3y + 5 = 0.$
35. Tangent: $x + y + 1 = 0.$
Normal: $x - y - 3 = 0.$
36. Tangent: $x + y - 2 = 0.$
Normal: $x - y = 0.$

37. $y = \pm x.$ **39.** $2a.$ **40.** $a.$

41. $y - 2x + 1 = 0, y - 6x + 9 = 0.$ **42.** 2.0650.

43. $-0.6328.$ **44.** 2.0393. **45.** $0°.$
46. $\tan^{-1}\frac{3}{4}, 90°.$ **47.** $\tan^{-1} 2.$ **50.** $5 + 3\sqrt{5}.$
52. $(-\frac{1}{3}, 0).$ **53.** $6x - 4y + 21 = 0.$ **54.** $y^2 = 4x - 4.$

55. $3x^2 + 3y^2 + 8x - 16 = 0.$ **56.** $x^2 - 3y^2 - 12y + 36 = 0.$
57. $x^2 + y^2 - 6x - 8y = 0.$ **58.** $x^2 + y^2 - 2x - 4y = 5.$
59. Center $(-3, 1)$, radius 4. **60.** Center $(2, -\frac{3}{2})$, radius 3.
63. $x^2 + y^2 - 6x + 2y - 40 = 0.$ **65.** $x^2 + y^2 - 2x - 2y + 1 = 0,$
$x^2 + y^2 - 10x - 10y + 25 = 0.$
66. $x^2 + y^2 = 1, x^2 + y^2 = 81.$ **67.** $x^2 + y^2 - 2y - 1 = 0.$
68. $x^2 + y^2 - 2x + 4y - 20 = 0.$ **69.** $x^2 + y^2 + 2x - 4y - 20 = 0.$

72. $x - 2y = 0.$
74. Axis $y = 0$, vertex $(0, 0)$, focus $(2, 0)$.
75. Axis $y = 0$, vertex $(0, 0)$, focus $(-\frac{3}{2}, 0)$.
76. Axis $y = 2$, vertex $(-1, 2)$, focus $(0, 2)$.
77. Axis $x = 0$, vertex $(0, 1)$, focus $(0, \frac{5}{2})$.
78. Axis $x = 3$, vertex $(3, -2)$, focus $(3, -1)$.
79. Axis $x = 6$, vertex $(6, 3)$, focus $(6, 0)$.

80. Focus $(0, 3)$, directrix $x = 4$. **81.** Focus $(-1, \frac{7}{2})$, directrix $y = \frac{1}{2}$.
83. 9 ft. **84.** 10 ft.
85. 35.56 ft. **86.** $\frac{2}{3}bh.$
87. $\dfrac{8\pi h^2 b}{15}.$ **88.** $2p.$

90. Center $(0, 0)$; axes $x = 0, y = 0$; foci $(\pm\sqrt{3}, 0)$; $e = \frac{1}{2}\sqrt{3}.$
91. Center $(0, 1)$; axes $x = 0, y = 1$; foci $(\pm\frac{1}{2}\sqrt{5}, 1)$; $e = \frac{1}{3}\sqrt{5}.$
92. Center $(-2, 1)$; axes $x = -2, y = 1$; foci $(-2 \pm \sqrt{3}, 1)$; $e = \frac{1}{2}\sqrt{3}.$
93. Center $(0, 0)$; axes $x = 0, y = 0$; foci $(0, \pm\sqrt{7})$; $e = \frac{1}{4}\sqrt{7}.$
94. Center $(1, -\frac{1}{2})$; axes $x = 1, y = -\frac{1}{2}$; foci $(\frac{1}{2}, -\frac{1}{2}), (\frac{3}{2}, -\frac{1}{2})$; $e = \frac{1}{2}.$
95. Center $(2, \frac{9}{2})$; axes $x = 2, y = \frac{9}{2}$; foci $(2, \frac{1}{2}[9 \pm 3\sqrt{5}])$; $e = \frac{1}{3}\sqrt{5}.$

Chapter IV (Continued)

96. $4x^2 + y^2 + 16x - 8y + 16 = 0$. **97.** $4x^2 + 3y^2 - 12y = 36$.
98. $x^2 + 4y^2 + 4x - 8y = 32$. **99.** $2x + 3y - 5 = 0$.

105. Center (0, 0); axes $x = 0$, $y = 0$; foci $(\pm\sqrt{13}, 0)$; $e = \frac{1}{2}\sqrt{13}$; asymptotes $2y = \pm 3x$.
106. Center (1, 2); axes $x = 1$, $y = 2$; foci $(1 \pm \sqrt{5}, 2)$; $e = \sqrt{5}$; asymptotes $y - 2 = \pm 2(x - 1)$.
107. Center $(2, -1)$; axes $x = 2$, $y = -1$; foci $(2, -1 \pm \sqrt{6})$; $e = \frac{1}{2}\sqrt{6}$; asymptotes $y + 1 = \pm\sqrt{2}(x - 2)$.
108. Center $(1, -3)$; axes $x = 1$, $y = -3$, foci $(1 \pm 2\sqrt{5}, -3)$; $e = \sqrt{2}$, asymptotes $y + 3 = \pm(x - 1)$.

109. $3y^2 - x^2 - 12y + 9 = 0$. **110.** $4(x - 2)^2 - (y + 1)^2 = \pm 4$.

111. $8x^2 - y^2 + 16x + 2y = 0$.
113. A hyperbola with foci at the rifle and the target.
115. Hyperbola through A with one focus at B. Eccentricity = 2.

118. $(-1, 5)$, $(-7, 7)$, $(2, -5)$. **119.** $x^2 + 4y^2 - 4 = 0$.
120. $y^3 + 3x^2 + 16 = 0$. **121.** $x + y = 0$, $2x + 3y = 0$.
122. $x^2 + y^2 = 11$. **123.** $(-\frac{1}{2}, -2)$.
124. $3x^2 + y^2 = 2$, or $x^2 + 3y^2 = 2$. **125.** $xy = 1$.
126. $xy = 2$. **127.** $\dfrac{\pi}{8}$.
128. $y^2 = \sqrt{2}(x - y) + 14$. **129.** $y^2 = ax\sqrt{2} - \frac{1}{2}a^2$.

Chapter V. Pages 156–158

1. $x = 1$, $y = -2$. **2.** $x = 2$, $y = 1$. **3.** $x = -2$, $y = 1$.
4. $x = 0$, $y = -1$. **5.** -18. **6.** 0.
7. 0. **8.** 42. **9.** 63.

10. 0. **11.** $x = \frac{5}{4}$, $y = -\frac{1}{4}$, $z = -\frac{1}{4}$.
12. $x = 6$, $y = -3$, $z = 10$. **13.** $x = 1$, $y = -1$, $z = 2$.
14. $x = 3$, $y = 4$, $z = 2$. **15.** $x = -1$, $y = 2$, $z = 1$, $u = 3$.
16. $x = 3$, $y = -1$, $z = -1$, $u = 0$. **19.** $x:y:z = 3:-11:-4$.
20. $x:y:z = 1:2:-3$. **21.** $x:y:z = 3:0:-2$.
22. $x:y:z = 2:-1:1$. **23.** $x:y:z = 3:-1:2$.

26. $\begin{vmatrix} y^2 & x & 1 \\ 9 & 2 & 1 \\ 4 & -3 & 1 \end{vmatrix} = 0$.
27. $9(x^2 + y^2) - 44x - 41y + 32 = 0$.

28. The points are not on a circle. **29.** Converse not true.
30. $x = \frac{1}{5}$, $y = \frac{3}{5}$. **31.** Converse not true.
32. $k_1:k_2:k_3 = 5:-3:-1$.

Chapter VI. Pages 171–174

15. $12 \cos 3x - 12 \sin 4x$. **16.** $3 \sin \dfrac{x}{2} \cos \dfrac{x}{2}$.
17. $-12 \cos 3x \sin 3x$. **18.** $\sin^2 x$.

Chapter VI (Continued)

19. $2 \sin^2 x$.
20. $\frac{1}{2} x \cos \frac{x}{2}$.
21. $3 \cos 3x - 3 \sin^2 x \cos x$.
22. $\sin^3 \frac{x}{3}$.
23. $5 \cos^3 5x$.
24. $x \sin^3 x$.
25. $6 \sec^2 3x$.
26. $12 \csc^2 (1 - 4x)$.
27. $\frac{\sec^2 x}{2\sqrt{\tan x}}$.
28. $\tan^2 x$.
29. $-3 \cot^2 3x$.
30. $\tan 2x \sec^2 2x$.
31. 0.
32. $4 \sec^2 x \tan x$.
33. $3(\sec x + \tan x)^3 \sec x$.
34. $2 \sec^4 2x$.
35. $\tan^2 \frac{x}{3} \sec^4 \frac{x}{3}$.
36. $2 \sec \frac{2x}{3} \tan^3 \frac{2x}{3}$.
37. $\sec x \tan^5 x$.
38. $-\cot^6 x$.
39. $-\frac{y}{x}$.
41. 45°.
42. 135°.
43. $\tan^{-1} \frac{1}{3}$, $\tan^{-1} 3$, $\tan^{-1} \frac{3}{5}$.
44. 5.
45. $\frac{3}{2}$.
46. 0.9673
47. $\frac{2A}{b}$.
48. 80π mi./min.
49. 400 ft./sec., 200 ft./sec.
50. $\sin^{-1} \frac{1}{3}\sqrt{3}$.
51. 7.794 in.
52. 120°.
53. $\frac{a}{\pi}$.
54. 45°.
59. $\frac{12}{\sqrt{1 - 16x^2}}$.
60. $-\frac{4}{\sqrt{4 - x^2}}$.
61. $\frac{1}{\sqrt{2ax - x^2}}$.
62. -1.
63. $\frac{1}{x\sqrt{x^2 - 1}}$.
64. $\cos^{-1} x$.
65. $\sin^{-1} x$.
66. $\cos^{-1} 2x$.
67. $\sqrt{\frac{a - x}{a + x}}$.
68. $\frac{2x}{(x^2 + 2)\sqrt{x^2 + 1}}$.
69. $4x\sqrt{1 - x^4}$.
70. $\frac{6}{4x^2 + 9}$.
71. $\frac{a}{x^2 + a^2}$.
72. $\frac{16}{(x^2 + 4)^2}$.
73. $\frac{2a}{a^2 + x^2}$.
74. $\frac{2a^3}{(x^2 + a^2)^2}$.
75. 2.
76. $\frac{2}{1 + x^2}$.
77. $\frac{\sqrt{x^2 - 1}}{x}$.
78. $\frac{3}{(2x - 1)\sqrt{x^2 - x}}$.
79. $\frac{\sqrt{x^2 - a^2}}{x}$.
80. $\frac{1}{2x\sqrt{x - 1}}$.
81. $\frac{1}{\sqrt{1 - x^2}}$.
82. $\frac{2a^2}{x^3\sqrt{x^2 - a^2}}$.
83. $\frac{-2}{x^2 + 1}$;
84. $\frac{4}{5 + 3 \cos x}$.
85. $\frac{1}{5 + 3 \cos x}$.
86. $\sin^{-1} x + C$.
87. $\sec^{-1} x + C$.
88. $\tan x + C$.
89. $\sec x + C$.

Chapter VI (Continued)

93. 12 ft.

94. $\frac{\pi}{30}$.

95. 120π radians per second.

96. $5t$ radians per second.

97. $\frac{d\theta}{dt} = 0,\ \frac{d^2\theta}{dt^2} = -A\omega^2$.

99. $2n\pi$.

100. Amplitude 5, period $\frac{\pi}{3}$.

101. Amplitude 5, frequency $\frac{1}{\pi}$.

102. Amplitude $\frac{3}{2}$, period $\frac{\pi}{2}$.

103. $\frac{2\pi}{k}$.

Chapter VII. Pages 184–186

10. $\frac{3}{3x+2}$.

11. $\frac{6(x+1)}{3x^2+6x+1}$.

12. $\frac{2}{x-2}$.

13. $\ln x$.

14. $\frac{2}{x^2-1}$.

15. $\frac{6}{4-9x^2}$.

16. $\cot x$.

17. $\csc x$.

18. $2 \sec 2x$.

19. $\frac{1}{\sqrt{x^2-a^2}}$.

20. $\frac{\cos^3 (x/3)}{3 \sin (x/3)}$.

21. $4 \csc 2x$.

22. $\frac{2 \ln x}{x}$.

23. $\frac{\log_{10} e}{x+5}$.

24. $\frac{1}{2}(e^x - e^{-x})$.

25. $6xe^{3x^2}$.

26. $-6e^{-3x}$.

27. $e^{\sin x} \cos x$.

28. $ex^{e-1} + e^x$.

29. $10^x \ln 10$.

30. $x^{n-1}n^x(n + x \ln n)$.

31. $-9xe^{-3x}$.

32. $4x^2e^{2x}$.

33. $\frac{12}{(e^{3x}+e^{-3x})^2}$.

34. $\frac{4}{e^{2x}-e^{-2x}}$.

35. $-(2 \sin 2x + 3 \cos 2x)e^{-3x}$.

36. $e^x \sin 2x$.

37. $\sec^{-1} x$.

38. $\tan^{-1} \frac{x}{2}$.

39. $\frac{1}{x \ln (ax)}$.

40. $\frac{2(1+y-2x)}{1-y+2x}$.

41. $-e^{x-y}$.

42. $\frac{y(1-x)}{x(y-1)}$.

43. $e^x \cos^2 y$.

44. $\frac{y(1-xe^{x-y})}{x(1-ye^{x-y})}$.

45. $-x^{-x}(1 + \ln x)$.

46. $(\tan x)^x(\ln \tan x + 2x \csc 2x)$.

47. $\frac{2a^2x}{(x^2+a^2)\sqrt{x^4-a^4}}$.

48. $\frac{12x^2+21x+12}{2(2x+3)^{\frac{1}{2}}(3x+2)^{\frac{3}{2}}}$.

49. $\frac{2x(x^4+4x^2-16)}{(x^2-4)^{\frac{1}{2}}(x^2+4)^{\frac{3}{2}}}$.

51. e^{mn}.

54. 3.

55. $a^x(\ln a)^n$.

57. 1% increase.

58. $27e^{-8}$.

59. $x = \frac{7}{4}\pi,\ y = -\frac{1}{2}\sqrt{2}e^{\frac{7}{4}\pi}$.

60. $e^{-\frac{1}{e}}$.

64. $\frac{1}{2} \ln (2x+3) + C$.

Chapter VII (Continued)

65. $\frac{1}{2}\ln(x^2+1)+C$.
66. $\frac{1}{3}e^{3x}+C$.
67. $\frac{1}{2}e^{x^2}+C$.
68. $4\ln 4$.
69. $8\pi\ln 2$.
70. Ce^{kt}, C and k being constants.
71. $y^2 = 4x$.
72. 3 min.

Chapter VIII. Pages 197–201

1. $2x+3y-7=0$.
2. $x=(y-2)^2$.
3. $(x-y)^2 = 8(x+y)$.
4. $x^2+4y^2=4$.
5. $x^2-4y^2=4$.
6. $y^2 = \dfrac{x^3}{4-x}$.
7. $(y-x)^3 = 2(y+x)^2$.
8. $2(x^2+y^2)-2xy\sqrt{2}=1$.
9. $x^2-y^2=1$.
10. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.
11. $y^2=4x^2(1-x^2)$.
12. $xy=1,\ |x| \leqq 1$.
13. $2x+3y=13,\ -1 \leqq x \leqq 5$.
14. $\ln(x^2+y^2) = 2\tan^{-1}\dfrac{x}{y}$.
15. The second locus is part of first.
17. $(3, \pm 4)$.
18. $(1, 1), (0, \frac{3}{2})$.
19. $x = a\cos\phi,\ y = a\sin\phi$.
20. $x = a\cos\phi,\ y = b\sin\phi$.
21. $x = a\sec\phi,\ y = b\tan\phi$.
22. $x = a\cos^3\phi,\ y = a\sin^3\phi$.
23. $x = \dfrac{t}{1+t^3},\ y = \dfrac{t^2}{1+t^3}$.
24. $x = \dfrac{2at^2}{1+t^2},\ y = \dfrac{2at^3}{1+t^2}$.
25. $x = \dfrac{p}{m^2},\ y = \dfrac{2p}{m}$.
26. $x = \dfrac{a}{(1-m)^2},\ y = \dfrac{am^2}{(1-m)^2}$.
27. $x = \pm\dfrac{a^2m}{\sqrt{a^2m^2+b^2}},\ y = \mp\dfrac{b^2}{\sqrt{a^2m^2+b^2}}$.
28. $x = \pm\dfrac{a^2m}{\sqrt{a^2m^2-b^2}},\ y = \pm\dfrac{b^2}{\sqrt{m^2a^2-b^2}}$.
29. t.
30. $\dfrac{\pi}{2}-\dfrac{t}{2}$.
31. $x = b\cos\phi,\ y = a\sin\phi;\ \dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$.
32. $x = a\cos\phi + a\phi\sin\phi$,
$y = a\sin\phi - a\phi\cos\phi$, where a is the radius of the circle.
33. $x = a\phi - b\sin\phi,\ y = a - b\cos\phi$.
34. $x = a\cos^3\phi,\ y = a\sin^3\phi;\ x^{\frac{2}{3}}+y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
35. $x = 2a\tan\phi,\ y = 2a\cos^2\phi;\ y = \dfrac{8a^3}{x^2+4a^2}$.
36. $4\pi a^2$.
37. $x = 2a\sin^2\phi,\ y = 2a\sin^2\phi\tan\phi;\ (x^2+y^2)x = 2ay^2$.
38. $x = a\sin\phi,\ y = a\tan\phi\,(1+\sin\phi);\ y^2 = \dfrac{x^2(a+x)}{a-x}$.
39. $x = -a\cos(\phi+B),\ y = b\sin(\phi-A)$.

Chapter VIII (Continued)

42. $\frac{2}{3}(2\sqrt{2} - 1)$.
43. 37.9.
44. $2\pi a$.
45. $\frac{3}{2}a$.
46. $\sqrt{2}(e^{\pi} - 1)$.
47. 5.
48. $2a$.
49. 8.
50. $\pm\sqrt{15}$.
51. $8a$.
52. $\dfrac{8a(a + b)}{b}$.
55. $\sec x$.
56. $\dfrac{(y^2 + 1)^2}{4y}$.
57. $a\theta$.
58. $\frac{3}{2}a \sin 2t$.
59. $4 \sin t$.
60. $a \sec^2 t$.
61. $\frac{13}{6}\sqrt{13}$.
62. $\dfrac{b^2}{a}$.
63. $\frac{5}{4}\sqrt{5}$.
64. ∞, 1.
65. $\dfrac{4a(a + b)}{2a + b} \sin\left(\dfrac{b\phi}{2a}\right)$.

Chapter IX. Pages 214–217

3. $\left(4, 2n\pi - \dfrac{\pi}{8}\right), \left(-4, 2n\pi + \dfrac{7\pi}{8}\right)$.
4. $(4, 360n - 300), (-4, 360n - 120)$.
33. $\left(a\sqrt{2}, \dfrac{\pi}{4}\right)$.
34. $\left(a, \pm\dfrac{\pi}{3}\right)$.
35. $\left(a\sqrt{2}, \dfrac{\pi}{2}\right)$.
36. $\left(\dfrac{a\sqrt{2}}{2}, 0\right), (0, 0)$.
37. $\left(a, \pm\dfrac{\pi}{6}\right), \left(a, \pm\dfrac{5\pi}{6}\right)$.
38. $\left[a\left(1 + \dfrac{1}{2}\sqrt{2}\right), \dfrac{\pi}{4}\right], \left[a\left(1 - \dfrac{1}{2}\sqrt{2}\right), \dfrac{5\pi}{4}\right], (0, 0)$.
39. $\left(a, \pm\dfrac{\pi}{4}\right), \left(a, \pm\dfrac{3\pi}{4}\right), (0, 0)$.
40. $\left(\pm a\sqrt[4]{\dfrac{3}{4}}, \dfrac{\pi}{3}\right), (0, 0)$.
49. $(x^2 + y^2)^2 - 2ax(x^2 + y^2) = a^2y^2$.
50. $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$.
51. $r = 4 \cot\theta \csc\theta$.
52. $r = 2\cos\theta$.
53. $r^2 \sin 2\theta = 2$.
54. $r^2 \cos 2\theta = 1$.
55. $r = 4a(\cos\theta + \sin\theta)$.
56. $r^2 = a^2 \cos 2\theta$.
57. $r = a \sec\theta$.
58. $r \cos\left(\theta - \dfrac{\pi}{3}\right) = 3$.
59. $r(b\cos\theta + a\sin\theta) = ab$.
60. $r = 2a\cos\theta$.
61. $r = 2a\sin\theta$.
62. $r = 4\cos\left(\theta - \dfrac{\pi}{6}\right)$.
63. $r = \dfrac{2a}{1 + \sin\theta}$.
64. $r = \dfrac{4}{1 - \cos\left(\theta - \dfrac{\pi}{6}\right)}$.
65. Circle.

Chapter IX (Continued)

66. $r = a\cos\theta + b$.
67. Circle.
68. Straight line.
69. $r = a\sin 2\theta$.
70. Lemniscate $r^2 = 2a^2\cos 2\theta$, if the points are $(\pm a, 0)$.
71. $\tan^{-1}\theta$.
74. $90°$.
75. $90°$.
76. $0°$.
77. $\tan^{-1} 2$.
78. $\tan^{-1}(\frac{3}{5}\sqrt{3})$, $0°$.
79. $\tan^{-1}(4\sqrt{3})$, $90°$.
82. $\dfrac{1-\cos\theta-2\cos^2\theta}{\sin\theta(1+2\cos\theta)}$.
83. $(\frac{3}{2}a, \pm\frac{2}{3}\pi)$.
86. $2\pi a$.
87. $\frac{4}{3}a\sqrt{3}$.
88. $\dfrac{\sqrt{1+k^2}}{k}(r-a)$.
89. $8a$.
90. $\frac{3}{2}\pi a$.
92. $\dfrac{a(\theta^2+1)^{\frac{3}{2}}}{\theta^2+2}$.
93. $\dfrac{4}{3}a\cos\dfrac{\theta}{2}$.
94. $a\sqrt{1+k^2}\,e^{k\theta}$.
95. $\dfrac{3}{4}a\sin^2\dfrac{\theta}{3}$.
96. $\frac{2}{3}a$.
97. a^2.
98. $\dfrac{a^2}{n}$.
99. $\dfrac{r_2^2-r_1^2}{4k}$.
100. $\frac{1}{2}k(r_1-r_2)$.
101. $\frac{3}{2}\pi a^2$.
102. $\frac{1}{3}a^2(3\sqrt{3}-\pi)$.

Chapter X. Pages 238–242

3. $\overrightarrow{DE} = \frac{2}{3}(\overrightarrow{AB} + \overrightarrow{BC})$.
4. $\overrightarrow{AE} = \mathbf{u} + \frac{1}{2}\mathbf{v}$, $\overrightarrow{CD} = -\frac{1}{2}\mathbf{u} - \mathbf{v}$, $\overrightarrow{DE} = \frac{1}{2}(\mathbf{u}+\mathbf{v})$.
5. $\overrightarrow{AD} = \frac{2}{3}\mathbf{B} + \frac{1}{3}\mathbf{C} - \mathbf{A}$.
6. $\overrightarrow{DE} = \frac{1}{2}\mathbf{C} - \frac{1}{3}\mathbf{B} - \frac{1}{6}\mathbf{A}$.
8. $\mathbf{v} + \frac{1}{2}\mathbf{u}$.
9. $\overrightarrow{CE} = \mathbf{u} - \frac{1}{3}\mathbf{v}$, $\overrightarrow{CF} = \frac{3}{4}\mathbf{u} - \frac{1}{4}\mathbf{v} = \frac{3}{4}\overrightarrow{CE}$.
10. $\mathbf{B}-\mathbf{A}$, $-\mathbf{A}$, $-\mathbf{B}$, $\mathbf{A}-\mathbf{B}$.
11. $\mathbf{A}_3 = 2\mathbf{A}_2\cos\dfrac{2\pi}{n} - \mathbf{A}_1$.
15. Same as 14.
18. $\dfrac{\mathbf{B}+m\mathbf{C}}{1+m}$.
19. $\dfrac{m\mathbf{A}+n\mathbf{B}+p\mathbf{C}}{m+n+p}$.
23. 2, $2\sqrt{3}$.
24. $\overrightarrow{AB} = -3\mathbf{i} + 7\mathbf{j}$, $\overrightarrow{BC} = 4\mathbf{i} + \mathbf{j}$, $\overrightarrow{CA} = -\mathbf{i} - 8\mathbf{j}$.
26. $\dfrac{b}{a}$.
29. $\frac{9}{5}$, $\frac{16}{5}$.
30. 1.
32. $x = -1$, $y = 3$.
33. (5, 0).
34. (1, 4).
36. 0, $2\mathbf{A}$.
37. $\mathbf{i} - \mathbf{j}$.
38. 0.
40. $\mathbf{C}_1 = 0$, $\mathbf{C}_2 = -\frac{1}{2}\mathbf{A}$.

Chapter X (Continued)

41. $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$, $\mathbf{a} = 2\mathbf{i}$, speed $= 5$.
42. $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j}$, speed $= \sqrt{13}$.
43. $\mathbf{v} = -2\mathbf{j}$, $\mathbf{a} = -3\mathbf{i}$, speed $= 2$.
44. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, speed $= 5$.
45. $\mathbf{v} = \mathbf{i}\sqrt{3} + \mathbf{j}$, $\mathbf{a} = -4\mathbf{i}$, speed $= 2$.
46. $\mathbf{v} = 2(\mathbf{i} - \mathbf{j})$, $\mathbf{a} = 4(\mathbf{i} + \mathbf{j})$, speed $= 2\sqrt{2}$.
47. $v_x = 24$, $v_y = 8 - 32t$, $a_x = 0$, $a_y = -32$, speed $= 8\sqrt{9 + (1 - 4t)^2}$.
48. $v_x = 1 + t$, $v_y = 1 - t$, $a_x = 1$, $a_y = -1$, speed $= \sqrt{2 + 2t^2}$.
49. $v_x = -t \sin t$, $v_y = t \cos t$, $a_x = -(t \cos t + \sin t)$, $a_y = \cos t - t \sin t$, speed $= |t|$.
50. $v_x = -\tan t$, $v_y = 1$, $a_x = -\sec^2 t$, $a_y = 0$, speed $= |\sec t|$.
54. $\mathbf{a} = \dfrac{2\mathbf{j}cu^2}{x^3}$.
55. $a_t = \dfrac{3 \cos t \sin t}{\sqrt{3 \sin^2 t + 1}}$, $a_n = \dfrac{2}{\sqrt{3 \sin^2 t + 1}}$.
56. $a_t = 2\sqrt{a^2 + b^2}$, $a_n = 0$.
57. $a_t = 0$, $a_n = k^2 a$.
58. $a_t = \dfrac{a(e^{2t} - e^{-2t})}{\sqrt{2(e^{2t} + e^{-2t})}}$, $a_n = \dfrac{2a}{\sqrt{2(e^{2t} + e^{-2t})}}$.
59. $a_t = ak^2 \cos \dfrac{\phi}{2}$, $a_n = ak^2 \sin \dfrac{\phi}{2}$.
60. $a_t = k^2 a$, $a_n = k^2 a\phi$.
62. $v_r = a\omega$, $v_\theta = a\omega^2 t$, $a_r = -\omega^3 at$, $a_\theta = 2a\omega^2$, speed $= a\omega\sqrt{1 + \omega^2 t^2}$.
63. $v_r = 2a(t + c)$, $v_\theta = ak(t + c)$, $a_r = a(2 - k^2)$, $a_\theta = 3ka$, speed $= a\sqrt{k^2 + 4}\,(t + c)$.
64. $v_r = a$, $v_\theta = -a\theta$, $a_r = -\dfrac{a^4k^2}{r^3}$, $a_\theta = 0$, speed $= a\sqrt{1 + \theta^2}$.
65. $v_r = \dfrac{a\omega}{2}(e^\theta + e^{-\theta})$, $v_\theta = \dfrac{a\omega}{2}(e^\theta - e^{-\theta})$, $a_r = 0$, $a_\theta = a\omega^2(e^\theta + e^{-\theta})$, speed $= a\omega\sqrt{\dfrac{e^{2\theta} + e^{-2\theta}}{2}}$.
66. $v_r = -kae^{-kt}$, $v_\theta = 2kabe^{kt}$, $a_r = k^2 r - \dfrac{4k^2b^2a^4}{r^3}$, $a_\theta = 0$, speed $= k\sqrt{a^2e^{-2kt} + 4a^2b^2e^{2kt}}$.
67. $a_r = -\omega^2 r$, $a_\theta = 2v\omega$.

Chapter XI. Pages 268–274

1. $\frac{2}{9}(3x + 2)^{\frac{3}{2}} + C$.
2. $\dfrac{1}{2(3 - 2x)} + C$.
3. $\dfrac{2}{b}\sqrt{a + bx} + C$.
4. $ax - \frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} + \frac{1}{2}x^2 + C$.
5. $-\sqrt{a^2 - x^2} + C$.
6. $\frac{2}{9}(x^3 + 1)^{\frac{3}{2}} + C$.
7. $\frac{4}{5}(x^{\frac{2}{3}} - 1)^{\frac{3}{2}} + C$.
8. $\dfrac{1}{3(2 - 3x^2)} + C$.

Chapter XI (Continued)

9. $\frac{1}{4}\ln(4x+3)+C.$
10. $\frac{1}{4}\ln(2x^2-1)+C.$
11. $\frac{1}{3}\ln(1+x^3)+C.$
12. $\ln(x^2+3x+2)+C.$
13. $\frac{1}{2}\sin(4x+1)+C.$
14. $-\frac{5}{3}\cos\left(\frac{3x-2}{5}\right)+C.$
15. $-\frac{1}{2}\cos(x^2+1)+C.$
16. $\frac{1}{2a}\sin(ax^2)+C.$
17. $2\tan\frac{\theta}{2}+C.$
18. $\frac{1}{3}\tan 3\theta+C.$
19. $-\frac{1}{2}\cot 2\theta+C.$
20. $-\frac{1}{2}\cot(x^2)+C.$
21. $-\frac{5}{3}\ln\cos\frac{3x+4}{5}+C.$
22. $\frac{1}{2}\ln\sin 2x+C.$
23. $\frac{1}{2}\ln\sin 2x+C.$
24. $\frac{1}{5}\sec 5\theta+C.$
25. $-\csc x+C.$
26. $\tan\theta+\sec\theta+C.$
27. $\csc\theta-\cot\theta+C.$
28. $\frac{1}{2}(\tan 2x+\sec 2x)+C.$
29. $\frac{1}{3}\ln(\sec 3x+\tan 3x)+C.$
30. $\frac{1}{3}\ln[\csc(3x+2)-\cot(3x+2)]+C.$
31. $\ln(\sec^2 x+\sec x\tan x)+C.$
32. $\theta+\tan\theta+2\ln(\sec\theta+\tan\theta)+C.$
33. $-\frac{1}{2}\ln(1+2\cot x)+C.$
34. $\ln(\ln x)+C.$
35. $\sin^{-1}\frac{t}{\sqrt{2}}+C.$
36. $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{3}}+C.$
37. $3\sin^{-1}\frac{x}{2}-2\sqrt{4-x^2}+C.$
38. $\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}+C.$
39. $\frac{1}{2\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}+C.$
40. $\frac{1}{\sqrt{6}}\tan^{-1}\frac{x\sqrt{6}}{3}+C.$
41. $\frac{1}{2}\sec^{-1}\frac{x}{2}+C.$
42. $\frac{1}{3}\sec^{-1}\frac{2x}{3}+C.$
43. $\sec^{-1}x\sqrt{5}+C.$
44. $\ln(x+\sqrt{x^2+4})+C.$
45. $\frac{1}{2}\ln(2x+\sqrt{4x^2+3})+C.$
46. $\frac{1}{\sqrt{2}}\ln(x\sqrt{2}+\sqrt{2x^2+5})+C.$
47. $\frac{1}{8}\ln\frac{x-4}{x+4}+C.$
48. $\frac{1}{12}\ln\frac{2x-3}{2x+3}+C.$
49. $\frac{1}{2\sqrt{6}}\ln\frac{2x-\sqrt{6}}{2x+\sqrt{6}}+C.$
50. $-\frac{1}{3}e^{-3x}+C.$
51. $-\frac{1}{2}e^{-2x}+C.$
52. $\frac{1}{2}(e^{2x}-e^{-2x})-2x+C.$
53. $-\frac{1}{2}e^{-x^2}+C.$
54. $2\sqrt{1-\cos\theta}+C.$
55. $\sin^{-1}\frac{\sin\theta}{\sqrt{2}}+C.$
56. $\frac{1}{2}\sin^{-1}\frac{x^2}{a^2}+C.$
57. $\frac{1}{2}\ln(1+e^{2x})+C.$
58. $\tan^{-1}(e^x)+C.$
59. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sin\theta}{\sqrt{2}}+C.$
60. $-\sin^{-1}(e^{-t})+C.$
61. $\frac{1}{4}\sin^4 x+C.$
62. $-\frac{1}{25}\cos^5 5x+C.$

Chapter XI (Continued)

63. $\theta + \sin^2\theta + C.$

64. $\frac{1}{3}\cos^3 x - \cos x + C.$

65. $\frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C.$

66. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$

67. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$

68. $\frac{1}{16}\sin^4 4\theta - \frac{1}{24}\sin^6 4\theta + C.$

69. $-\cos x - \frac{1}{2}\cos^2 x + C.$

70. $\ln(\csc x - \cot x) + \cos x + C.$

71. $\ln\sin x - \frac{1}{2}\sin^2 x + C.$

72. $\frac{1}{3}\tan^3 x + C.$

73. $\frac{1}{4}\tan^4 x + C.$

74. $\frac{1}{2}\tan^2 x + C.$

75. $\frac{1}{3}\tan^3 x + C.$

76. $\tan x + \frac{1}{3}\tan^3 x + C.$

77. $\tan x - x + C.$

78. $2\ln\sin\theta - \cot\theta + C.$

79. $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C.$

80. $-\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C.$

81. $-\frac{1}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C.$

82. $\ln(\sec\theta + \tan\theta) + C.$

83. $\frac{1}{2}x + \frac{1}{16}\sin 8x + C.$

84. $\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C.$

85. $\frac{5}{2}x + 4\cos x - \frac{3}{4}\sin 2x - \frac{1}{3}\cos^3 x + C.$

86. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

87. $\frac{1}{8}x - \frac{1}{64}\sin 8x + C.$

88. $\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta + C.$

89. $\frac{5}{16}x + \frac{1}{4}\sin 2x - \frac{1}{48}\sin^3 2x + \frac{3}{64}\sin 4x + C.$

90. $-\frac{2}{3}\cos^3 x + C.$

91. $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C.$

92. $\frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C.$

93. $\frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C.$

94. $-\frac{1}{4}\sin x + \frac{1}{6}\sin 3x - \frac{1}{28}\sin 7x + C.$

95. $\ln(\csc 2x - \cot 2x) + C.$

96. $-\cot\dfrac{x}{2} + C.$

97. $2\sqrt{2}\sin\dfrac{\theta}{2} + C.$

98. $4\sqrt{2}\left(\sin\dfrac{\theta}{2} - \dfrac{1}{3}\sin^3\dfrac{\theta}{2}\right) + C.$

99. $\sin^{-1}\dfrac{x-1}{\sqrt{3}} + C.$

100. $\dfrac{1}{2}\sin^{-1}\dfrac{2x-1}{\sqrt{2}} + C.$

101. $\dfrac{1}{\sqrt{3}}\sin^{-1}\dfrac{(x-1)\sqrt{3}}{\sqrt{5}} + C.$

102. $\dfrac{1}{2}\tan^{-1}\dfrac{x+3}{2} + C.$

103. $\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x-1}{\sqrt{2}} + C.$

104. $\dfrac{1}{2}\sec^{-1}\dfrac{x-1}{2} + C.$

105. $\dfrac{1}{\sqrt{5}}\sec^{-1}\dfrac{2x-3}{\sqrt{5}} + C.$

106. $\sec^{-1}(2x-1) + C.$

107. $\frac{3}{4}\ln(x+3) + \frac{5}{4}\ln(x-1) + C.$

108. $\frac{1}{2}\ln(x^2 + 2x - 3) + C.$

109. $\frac{1}{4}\ln(2x^2 - 2x + 1) - \frac{1}{2}\tan^{-1}(2x-1) + C.$

110. $\frac{1}{2}\ln(x^2 + 2x + 2) - \tan^{-1}(x+1) + C.$

111. $\dfrac{3}{2}\sin^{-1}\dfrac{2x-1}{\sqrt{2}} - \sqrt{1 + 4x - 4x^2} + C.$

112. $3\sqrt{x^2 + 2x + 3} - 5\ln(x + 1 + \sqrt{x^2 + 2x + 3}) + C.$

113. $\dfrac{2}{3}\sqrt{3x^2 - 6x - 1} + \dfrac{1}{\sqrt{3}}\ln[\sqrt{3}(x-1) + \sqrt{3x^2 - 6x - 1}] + C.$

114. $-\dfrac{1}{\sqrt{x^2 - 2x + 3}} + C.$

115. $\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{e^x + 1}{\sqrt{2}} + C.$

Chapter XI (Continued)

116. $x + \frac{4}{5} \ln (x - 2) - \frac{9}{5} \ln (x + 3) + C.$
117. $2 \ln x - \ln (x + 1) + C.$
118. $\frac{1}{2}x^2 + 4x - 2 \ln (x - 1) + 12 \ln (x - 2) + C.$
119. $\frac{1}{2} \ln (x^2 - 1) - \ln x + C.$
120. $4 \ln (x + 1) - \frac{3}{2} \ln x - \frac{5}{2} \ln (x + 2) + C.$
121. $x + \dfrac{1}{x} + 2 \ln (x - 1) - \ln x + C.$

122. $\dfrac{1}{x + 1} + \ln (x + 1) + C.$

123. $\ln (x - 2) - \dfrac{4}{x - 2} + C.$

124. $2 \ln \dfrac{x}{x - 1} - \dfrac{5}{x - 1} + C.$

125. $\dfrac{2}{x} + \dfrac{2}{x^2} + \ln \dfrac{x - 2}{x} + C.$

126. $\dfrac{1}{4}\left(\ln \dfrac{x + 1}{x - 1} - \dfrac{2x}{x^2 - 1}\right) + C.$

127. $\ln \dfrac{x}{\sqrt{1 + x^2}} - x + \tan^{-1} x + C.$

128. $\ln \dfrac{\sqrt{x^2 + 1}}{x + 1} + C.$

129. $\ln \dfrac{x^2 - 1}{x^2 + 1} + C.$

130. $\ln \dfrac{(x - 1)^2}{x^2 + x + 1} + C.$

131. $x + \dfrac{1}{8} \ln \dfrac{x^2 - 2}{x^2 + 2} + \dfrac{1}{2\sqrt{2}} \ln \dfrac{x - \sqrt{2}}{x + \sqrt{2}} - \dfrac{1}{\sqrt{2}} \tan^{-1} \dfrac{x}{\sqrt{2}} + C.$

132. $\sqrt{2} \tan^{-1} \dfrac{x}{\sqrt{2}} - \tan^{-1} x + C.$

133. $\tan^{-1} x - \dfrac{1}{2} \tan^{-1} \dfrac{x}{2} + C.$

134. $\dfrac{1}{2\sqrt{2}} \tan^{-1} \dfrac{x - 1}{\sqrt{2}} - \dfrac{1}{2} \tan^{-1} x + C.$

135. $\dfrac{1}{2} \ln (x^2 + 4) + \dfrac{2}{x^2 + 4} + C.$

136. $\ln x + \dfrac{1}{x^2 + 1} + C.$

137. $\dfrac{1}{\sqrt{2}} \tan^{-1} \dfrac{x - 1}{\sqrt{2}} - \dfrac{1}{x^2 - 2x + 3} + C.$
138. $-[x + 4\sqrt{x} + 4 \ln (1 - \sqrt{x})] + C.$
139. $\frac{2}{3}(x + 1)^{\frac{3}{2}} - 2(x + 1)^{\frac{1}{2}} + C.$
140. $\frac{2}{3}a(x - a)^{\frac{3}{2}} + \frac{2}{5}(x - a)^{\frac{5}{2}} + C.$
141. $2\sqrt{x + 2} - 2 \tan^{-1} \sqrt{x + 2} + C.$
142. $2 \tan^{-1} \sqrt{x - 1} + C.$
143. $\dfrac{2}{1 + \sqrt{x - 1}} + 2 \ln (1 + \sqrt{x - 1}) + C.$
144. $\frac{4}{3}x^{\frac{3}{4}} - 4x^{\frac{1}{4}} + 4 \tan^{-1} (x^{\frac{1}{4}}) + C.$
145. $2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} + 6 \ln (x^{\frac{1}{6}} - 1) + C.$

Chapter XI (Continued)

146. $\frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}} - \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C.$

147. $-\frac{1}{3}(x^2+2a^2)\sqrt{a^2-x^2}+C.$

148. $-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$

149. $\frac{1}{a}\ln\left(\frac{a-\sqrt{a^2-x^2}}{x}\right)+C.$

150. $-\frac{\sqrt{a^2-x^2}}{a^2x}+C.$

151. $\frac{x}{a^2\sqrt{a^2-x^2}}+C.$

152. $\frac{\sqrt{x^2-a^2}}{a^2x}+C.$

153. $\frac{1}{2a^3}\left(\sec^{-1}\frac{x}{a}+\frac{a\sqrt{x^2-a^2}}{x^2}\right)+C.$

154. $\sqrt{x^2-a^2}-a\sec^{-1}\frac{x}{a}+C.$

155. $\frac{1}{2a}\left(\sec^{-1}\frac{x}{a}-\frac{a\sqrt{x^2-a^2}}{x^2}\right)+C.$

156. $\frac{(a^2+2x^2)\sqrt{x^2-a^2}}{3a^4x^3}+C.$

157. $\frac{1}{15}(3x^2-2a^2)(x^2+a^2)^{\frac{3}{2}}+C.$

158. $-\frac{\sqrt{a^2+x^2}}{a^2x}+C.$

159. $\frac{x}{a^2\sqrt{a^2+x^2}}+C.$

160. $\ln(x+\sqrt{a^2+x^2})-\frac{x}{\sqrt{a^2+x^2}}+C.$

161. $\frac{1}{2a}\left(\tan^{-1}\frac{x}{a}-\frac{ax}{x^2+a^2}\right)+C.$

162. $x\sin x+\cos x+C.$

163. $\sin x - x\cos x + C.$

164. $x\tan x+\ln\cos x+C.$

165. $x\sec x-\ln(\sec x+\tan x)+C.$

166. $(x^2-2x+2)e^x+C.$

167. $\frac{1}{25}x^5(5\ln x-1)+C.$

168. $\frac{1}{2}(x^2-1)e^{x^2}+C.$

169. $x\sin^{-1}x+\sqrt{1-x^2}+C.$

170. $x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+C.$

171. $x\ln(x+\sqrt{x^2+a^2})-\sqrt{x^2+a^2}+C.$

172. $\frac{1}{2}x^2\sec^{-1}x-\frac{1}{2}\sqrt{x^2-1}+C.$

173. $2(x-1)\sin x+(1+2x-x^2)\cos x+C.$

174. $\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln(x+\sqrt{x^2-a^2})+C.$

175. $\frac{1}{2}x\sqrt{x^2+a^2}+\frac{1}{2}a^2\ln(x+\sqrt{x^2+a^2})+C.$

176. $\frac{1}{2}x\sqrt{x^2-a^2}+\frac{1}{2}a^2\ln(x+\sqrt{x^2-a^2})+C.$

177. $\frac{1}{13}e^{2x}(2\sin 3x-3\cos 3x)+C.$

178. $\frac{1}{2}e^{-x}(\sin x-\cos x)+C.$

Chapter XI (Continued)

179. $-\frac{1}{5}(3 \cos 3x \cos 2x + 2 \sin 3x \sin 2x) + C.$
180. $\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln (\sec x + \tan x) + C.$

181. $\frac{5}{32}\pi a^6.$	**182.** $\dfrac{x(3a^2 - 2x^2)}{3a^4(a^2 - x^2)^{\frac{3}{2}}} + C.$	**183.** 3.
184. Divergent.	**185.** Divergent.	**186.** 10.
187. 2.	**188.** $\frac{1}{3}.$	**189.** Divergent.
190. Divergent.	**191.** Convergent.	**192.** Convergent.
193. Divergent.	**194.** Convergent.	**195.** Divergent.
196. Convergent.	**197.** Divergent.	**198.** Convergent.
204. 2.0392.	**205.** 8.040.	**206.** 9.079.
207. 2.285.	**208.** 1.760.	**209.** 0.836.

Chapter XII. Pages 293–299

1. $\pi a\sqrt{a^2 + h^2}.$	**2.** $\frac{8}{3}\pi p^2(2\sqrt{2} - 1).$	**3.** $\frac{49}{9}\pi.$
4. $2\pi ah.$	**5.** $\frac{61}{6}\pi.$	**6.** $\frac{61}{432}\pi.$
7. $2\pi a(x_2 - x_1).$	**8.** $\frac{32}{5}\pi a^2.$	**9.** $\pi^2 a^2.$
10. $4\pi a^2(2 - \sqrt{2}).$	**11.** $4\pi a^2\sqrt{2}.$	**12.** $\frac{12}{5}\pi a^2.$
13. $\frac{64}{3}\pi a^2.$	**14.** $\pi a^2[\sqrt{3} + \frac{1}{2}\sqrt{2} \ln (\sqrt{2} + \sqrt{3})].$	
15. $6\pi^2 a^2.$	**16.** $\frac{4}{3}p^2(2\sqrt{2} - 1).$	**17.** $2a^2.$
18. $2k\pi^2 a^2.$	**19.** $16a^2.$	**20.** $a^2(\sqrt{3} - \frac{1}{3}\pi).$
21. $8a^2.$	**22.** $16a^2.$	**23.** $\dfrac{2}{\pi}.$
24. $\frac{1}{2}.$	**25.** 0.	**26.** $\dfrac{\pi a}{4}, \dfrac{2a}{\pi}.$
27. $\frac{1}{2}gt.$	**28.** $\frac{2}{3}gt.$	**29.** $2a^2.$
30. $\frac{2}{3}\pi a^2.$	**31.** $\frac{2}{3}k^2a^2.$	**32.** $\frac{1}{2}RA^2, I = \frac{1}{2}A\sqrt{2}$
33. 37.6 lb./in.^{2}	**34.** $\frac{1}{2}pb^2h$ lb-ft.	**35.** $\frac{1}{12}wbh^3.$
36. 70,312,000 lb-ft.	**37.** $\frac{81}{4}\pi w$ lb-ft.	**38.** $\frac{1}{3}h.$
39. $5\frac{1}{3}$ ft.	**40.** (2.4, 0).	**41.** $\left(0, \dfrac{4b}{3\pi}\right).$
42. $\left(\dfrac{\pi}{2}, \dfrac{\pi}{8}\right).$	**43.** $(\frac{1}{5}a, \frac{1}{5}a).$	

44. $9\frac{9}{28}$ in. from 12-in. edge, $2\frac{3}{7}$ in. from 24-in. edge.

45. Distance from center $= \dfrac{4(a^2 + ab + b^2)}{3\pi(a + b)}.$

46. $(\frac{9}{20}a, \frac{9}{20}a).$	**47.** $(\frac{1}{2}, \frac{3}{5}).$
48. $(\frac{21}{20}, -\frac{57}{70}).$	**49.** $(0, \frac{2}{5}a).$
50. $\bar{x} = \dfrac{3\sqrt{3}}{2\pi}, \bar{y} = \dfrac{3}{2\pi}.$	**51.** $\bar{x} = \pi a, \bar{y} = \frac{5}{6}a.$
52. $(\pi a, \frac{4}{3}a).$	**53.** $\bar{x} = 0, \bar{y} = \frac{5}{6}a.$
54. $\bar{x} = 0, \bar{y} = \frac{4}{5}a.$	

55. On the bisector at distance $\dfrac{2}{3}a\dfrac{\sin \alpha}{\alpha}$ from the center of the circle.

Chapter XII (Continued)

56. On the diameter that bisects the arc at distance $a\dfrac{\sin\alpha}{\alpha}$ from the center of the circle.

57. $\bar{x} = -\frac{17}{18}a,\ \bar{y} = 0.$ **58.** $\bar{x} = \frac{44}{75}a,\ \bar{y} = 0.$

59. $\bar{x} = \frac{1}{4}\pi a,\ \bar{y} = 0.$

60. On the axis at distance $\frac{1}{4}h$ from the base.

61. On the axis at distance $\frac{1}{3}h$ from the base.

62. $(\frac{2}{3}a, 0).$ **63.** $(0, \frac{5}{12}a).$

64. $(0, \frac{3}{8}b).$ **65.** $(\frac{1}{2}a, 0).$

66. On the radius perpendicular to its plane face at distance $\dfrac{3(b^4 - a^4)}{8(b^3 - a^3)}$ from the center.

67. The midpoint of the radius perpendicular to its plane face.

68. At height k. **69.** At height $\dfrac{b}{m+2}$.

70. At depth $\frac{2}{3}h$. **71.** At depth $\frac{1}{2}h$.

72. $\frac{1}{3}wbh^2.$ **74.** $2\pi^2a^2b.$

75. At distance $2r/\pi$ from the center of the circle.

76. $\frac{1}{4}\pi(6\sqrt{3} - 1).$ **77.** $\frac{1}{12}\pi(9\pi - 2).$

78. $(\pi a, \frac{4}{3}a)$. Area $= \frac{32}{3}\pi a^2.$ **79.** $16\pi^2a^2.$

80. $\frac{2}{3}\pi a^3(\cos\beta - \cos\alpha).$ **81.** $\theta = 0,\ r = \frac{3}{8}a(\cos\alpha + \cos\beta).$

82. $\frac{40}{3}\pi a^3.$ **83.** $\frac{1}{3}\pi a^3[3\ln(1 + \sqrt{2}) - \sqrt{2}].$

84. $\frac{1}{3}ab^3.$ **85.** $\frac{1}{4}bh^3.$ **86.** $\frac{1}{4}Ma^2.$

87. $\frac{1}{4}\pi ab^3.$ **88.** $\frac{1}{3}Ml^2.$ **89.** $\frac{1}{2}Ma^2.$

90. $\frac{1}{2}M(r_1^2 + r_2^2).$ **91.** $\frac{8}{15}\pi ab^4.$ **92.** $\frac{256}{9}\pi.$

93. $\frac{1}{2}Ma^2.$ **94.** $\frac{1}{5}a\sqrt{10}.$ **95.** $\frac{1}{3}a\sqrt{6}.$

96. $\sqrt{b^2 + \frac{3}{4}a^2}.$ **97.** $\frac{1}{4}Ma^2\omega^2.$ **98.** $\frac{1}{5}Ma^2\omega^2.$

Chapter XIII. Pages 319–325

1. $S = 1.$

2. $u_1 = 1,\ u_n = \dfrac{1}{n^2} - \dfrac{1}{(n-1)^2},\ S = 0.$

3. $S_n = 1 - \dfrac{1}{n+1},\ S = 1.$ **4.** $S = \dfrac{1}{1+a}.$

5. Divergent. **6.** Convergent.

7. Divergent. **8.** Divergent.

9. $S_n = \frac{3}{2}(1 - 3^{-n}),\ S = \frac{3}{2}.$ **10.** $S_n = \frac{4}{5}[1 - (-4)^{-n}],\ S = \frac{4}{5}.$

11. $S_n = \dfrac{1 - e^{-nx}}{1 - e^{-x}},\ S = \dfrac{1}{1 - e^{-x}}.$

12. Divergent. **13.** Convergent.

14. Divergent. **15.** Convergent.

16. Divergent. **17.** Divergent.

18. Convergent. **19.** Convergent.

Chapter XIII (Continued)

20. Convergent if $p > 1$, divergent if $p \leqq 1$.
21. Convergent if $p > 1$, divergent if $p \leqq 1$.

22. Convergent. **23.** Convergent. **24.** Divergent.
25. Convergent. **26.** Convergent. **27.** Divergent.
28. Divergent. **29.** Divergent. **30.** Convergent.
31. Convergent. **32.** Convergent. **33.** Convergent.
34. Divergent. **35.** Convergent. **36.** Convergent.
37. Convergent. **38.** Convergent. **39.** Convergent.
40. Divergent. **41.** Convergent. **42.** Divergent.

43. About 400,000,000. **47.** $-1 \leqq x < 1$.
48. $-2 < x \leqq 2$. **49.** Convergent for all x.

50. $-1 \leqq x \leqq 1$. **51.** $-1 < x < 1$. **52.** $-1 \leqq x \leqq 1$.
53. $-1 < x < 1$. **54.** $|x| < \frac{1}{2}\sqrt{2}$. **55.** $-1 \leqq x \leqq 1$.
56. $-1 < x < 1$. **57.** $-1 < x < 3$. **58.** $-2 \leqq x < 0$.

59. $-5 < x < 1$. **60.** Convergent for all x.

61. $\frac{5}{3} < x < \frac{7}{3}$. **62.** $|x| > 1$. **63.** $|x - 1| > 1$.

64. $x + x^2 + x^3 + \cdots$. **65.** $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots$.

66. $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$. **67.** $1 + \dfrac{1}{2}x - \dfrac{1}{2\cdot 4}x^2 + \cdots$.

68. $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$. **69.** $-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$.

70. $-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \cdots$. **71.** $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \cdots$.

72. $x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{x^5}{5} + \cdots$. **73.** $x + x^2 + \frac{1}{3}x^3 + \cdots$.

74. $e^2 + e^2(x-2) + \dfrac{e^2(x-2)^2}{2!} + \dfrac{e^2(x-2)^3}{3!} + \cdots$.

75. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$.
76. $2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 - \cdots$.
77. $2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \cdots$.
78. $-1 - (x+1) - (x+1)^2 - (x+1)^3 - \cdots$.
79. $8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$.
80. $1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \cdots$.

81. $\dfrac{1}{2} + \dfrac{1}{2}\sqrt{3}\left(x - \dfrac{\pi}{6}\right) - \dfrac{1}{4}\left(x - \dfrac{\pi}{6}\right)^2 - \dfrac{1}{12}\sqrt{3}\left(x - \dfrac{\pi}{6}\right)^3 + \cdots$.

82. $-\left(x - \dfrac{\pi}{2}\right) + \dfrac{1}{3!}\left(x - \dfrac{\pi}{2}\right)^3 - \dfrac{1}{5!}\left(x - \dfrac{\pi}{2}\right)^5 + \dfrac{1}{7!}\left(x - \dfrac{\pi}{2}\right)^7 - \cdots$.

83. $3 - 2(x+2) + \frac{1}{3}(x+2)^2 + \frac{1}{18}(x+2)^3 + \cdots$.

84. 0.0872. **85.** 0.0875. **86.** 0.182.
87. 1.887. **88.** 0.933. **89.** 1.649.
90. 0.833. **91.** 0.100. **99.** 0.00019.
101. $x < 0.0316$. **102.** $x < 0.413$. **103.** $x < 0.66$.

104. 0.693. **105.** $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots$.

Chapter XIII (Continued)

111. $x - \frac{1}{3}x^3 + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \cdots.$

112. 0.493. **113.** 0.464. **114.** 2.004.

115. $1 - x + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots.$

117. $1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4 + \frac{61}{6!}x^6 + \cdots$

118. $1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots.$

119. $\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 - \frac{1}{40}(x-1)^5 + \cdots.$

Chapter XIV. Pages 345–350

1. i. **2.** i.
3. $2 - 11i$. **4.** $18 - i$.

5. $9 + 14i,\ 9 - 14i$.

11. $\overrightarrow{AB} = 2i + 2,\ \overrightarrow{BC} = 3i - 1,\ \overrightarrow{AC} = 5i + 1.$

14. $2(\cos\pi + i\sin\pi)$. **15.** $3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\cdot$

16. $2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\cdot$ **17.** $4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\cdot$

18. $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\cdot$ **19.** $6\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdot$

20. $2i\sqrt{3} - 2$. **21.** -8.
22. $1 + i\sqrt{3}$. **23.** $-3 + i\sqrt{3}$.
24. $\pm(1 + i)$. **25.** $\pm(\sqrt{3} + i)$.
26. $\pm\frac{1}{2}\sqrt{2}(\sqrt{3} - i)$. **27.** $\pm(1.0987 + 0.4551i)$.

29. $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$
30. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta,$
$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$

31. $z = -2,\ 1 \pm i\sqrt{3}$. **32.** $z = \frac{1}{2}(-1 \pm i\sqrt{3})$.
33. $z = \pm 2,\ \pm 2i$. **34.** $z = \pm i,\ \pm i\sqrt{5}$.

35. $z = \frac{1}{2}(\sqrt{3} \pm i),\ -\frac{1}{2}(\sqrt{3} \pm i)$.
36. $z = \frac{1}{2}(\sqrt{3} \pm i),\ -\frac{1}{2}(\sqrt{3} \pm i),\ \pm i$.

39. $|z| = 2$. **40.** $|z - 3| = 1$. **41.** $|z - 2| = 3$.
42. All values. **43.** $|z + 2| = 1$. **44.** $z = 0$.
45. $|z - 2| = 1$. **46.** All values. **53.** -1.
54. $\frac{1}{2} - \frac{1}{2}i\sqrt{3}$. **55.** $-e^2 i$. **56.** $0.5403 + 0.8415i$.

57. $r = \sqrt{5},\ \theta = \tan^{-1} 2$.
58. Point where circle touches x-axis. $v = \omega(\pm ai - a)$.
59. $C = z_0 + \frac{i}{\omega}v$, where z_0 is the center of the wheel.
60. Intersection of CB and the line perpendicular to CA at A.
62. $z_0 + \frac{ia\omega^2}{\omega^2 - i\alpha}$, where z_0 is the lowest point on the rolling wheel.

Chapter XIV (Continued)

63. Center of rotation at point of contact of the two wheels; center of acceleration on CO at distance $\dfrac{a^2}{a+b}$ from C.

65. $e^{i\omega t}[a + i(v - a\omega)t]$.

67. $\ln z + 2\pi i$.

69. $\frac{1}{3} \ln 2 + \frac{1}{9}\pi\sqrt{3}$.

71. $\cosh x = \frac{5}{3}$, $\tanh x = \frac{4}{5}$, $\coth x = \frac{5}{4}$, $\operatorname{sech} x = \frac{3}{5}$, $\operatorname{csch} x = \frac{3}{4}$.

72. $\sinh x = \pm\frac{3}{4}$, $\tanh x = \pm\frac{3}{5}$, $\coth x = \pm\frac{5}{3}$, $\operatorname{sech} x = \frac{4}{5}$, $\operatorname{csch} x = \pm\frac{4}{3}$.

73. $\sinh x = \frac{5}{12}$, $\cosh x = \frac{13}{12}$, $\coth x = \frac{13}{5}$, $\operatorname{sech} x = \frac{12}{13}$, $\operatorname{csch} x = \frac{12}{5}$.

74. $\sinh x = \pm 0.75$, $\cosh x = 1.25$, $\tanh x = \pm 0.6$, $\coth x = \pm 1.67$, $\operatorname{csch} x = \pm 1.33$.

75. $\frac{35}{12}$.

76. $\frac{24}{25}$.

90. $\frac{1}{3} \cosh^3 x - \cosh x + C$.

91. $x - \tanh x + C$.

92. $-\operatorname{sech}^{-1} x + C$.

93. $-\operatorname{csch}^{-1} x + C$.

94. 0.8814.

95. -0.1059.

96. 0.2747.

97. 0.2682.

Chapter XV. Pages 389–397

1. $\sqrt{y^2 + z^2}$.

2. $(0, y, 0)$.

3. $\sqrt{6}$.

4. $\frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$.

5. $\sqrt{13}$.

8. $(5, 2, 5)$.

10. $\frac{1}{3}(16\mathbf{i} - 5\mathbf{j} + 4\mathbf{k})$.

14. (a) $-\frac{1}{2}x^2$, (b) $\frac{1}{2}x^2$.

19. $\mathbf{j} - \mathbf{k}$.

20. $\mathbf{j} + \mathbf{k}$.

21. -1.

22. $-\frac{2}{3}$.

24. $\cos^{-1} \frac{2}{7}$, $\cos^{-1} \frac{3}{7}$, $\cos^{-1} \frac{6}{7}$.

25. 45°, 135°.

26. 60°.

27. 30°.

28. $A = 30°$, $B = 120°$, $C = 30°$.

29. $A = 135°$, $B = \cos^{-1} \dfrac{2}{\sqrt{5}}$, $C = \cos^{-1} \dfrac{3}{\sqrt{10}}$.

30. $x = \frac{1}{3}(x' + 2y' + 2z')$,
$y = \frac{1}{3}(2x' + y' - 2z')$,
$z = \frac{1}{3}(-2x' + 2y' - z')$.

31. $\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$.

32. 6.

33. $3\frac{1}{2}$.

34. 5.

38. $\mathbf{N} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, distance $= \frac{5}{3}$.

39. $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$.

40. $\frac{1}{3}(\mathbf{i} - 5\mathbf{j} + 4\mathbf{k})$.

41. $(\frac{1}{10}, 1, \frac{3}{10})$.

42. $\frac{1}{6}\sqrt{6}$.

43. $\frac{22}{7}$.

44. $\mathbf{M} = \mathbf{i}(cy - bz) + \mathbf{j}(az - cx) + \mathbf{k}(bx - ay)$, $M_x = cy - bz$.

45. $7\mathbf{k} - 8\mathbf{i} - 3\mathbf{j}$.

46. -18.

47 $\frac{2}{7}\sqrt{14}$.

50. 2.

51. 4.

52. $\frac{1}{6}$.

53. $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$.

54. $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \times \mathbf{D} = 0$.

55. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

56. $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = 0$.

69. $\mathbf{V} = \dfrac{l \mid \mathbf{A} \mid \mathbf{B} \times \mathbf{C} + m \mid \mathbf{B} \mid \mathbf{C} \times \mathbf{A} + n \mid \mathbf{C} \mid \mathbf{A} \times \mathbf{B}}{\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}}$.

Chapter XV (Continued)

60. $\dfrac{39\mathbf{i} + 18\mathbf{j} + 15\mathbf{k}}{11}$.

67. $x^2 + y^2 + z^2 - 4x - 2z = 4$.
68. $x^2 + z^2 = r^2$.
69. $z - y = 1$.
70. $y = 0,\ z = \pm x$.
72. $x^3 = 8z$.
73. $2x + 3y - z = 3$.
74. $x + y\sqrt{2} + z = 0$.
76. $3x - y + 2z + 1 = 0$.
77. $y - x = 1$.
80. $60°$.
81. $\cos^{-1} \frac{1}{3}$.
82. $45°$.
83. $\dfrac{x-2}{2} = \dfrac{y+1}{3} = \dfrac{z-3}{-4}$.
84. $x - 2 = y - 3 = \dfrac{z+1}{3}$.
85. $\dfrac{x-3}{4} = \dfrac{y-1}{-2} = z + 2$.
86. $\dfrac{x-2}{4} = \dfrac{y-1}{-3} = z + 3$.
87. $3x - z = 0$.
88. $2x - 3z = 2$.
89. $30°$.
90. $60°$.
91. $45°$.
96. $y^2 + z^2 = 4px$.
97. $y^2(x^2 + z^2) = 1$.
98. $\dfrac{x^2}{a^2} + \dfrac{y^2 + z^2}{b^2} = 1$.
99. $4(x^2 + z^2) = 9(y - 2)^2$.
100. $(z - 3)^2 = (x - 1)^2 + (y - 2)^2$.
101. $(z - 1)^2 + (r - 2)^2 = 16$.
116. $2\pi\sqrt{2}$.
117. 12.
118. 50.
119. $\sqrt{2} \ln (1 + \sqrt{2})$.
120. $\mathbf{F} \times \dfrac{d^2\mathbf{F}}{dt^2}$.
121. $\mathbf{F} \cdot \dfrac{d\mathbf{F}}{dt} \times \dfrac{d^3\mathbf{F}}{dt^3}$.
122. $x - 2 = \dfrac{y-4}{4} = \dfrac{z-8}{12}$.
123. $x = \dfrac{y-2}{3} = \dfrac{z-1}{4}$.
124. $\tan^{-1} \dfrac{a}{m}$.

125. $\mathbf{t} = \dfrac{\mathbf{i} + \mathbf{j}\cos x - \mathbf{k}\sin x}{\sqrt{2}}$, $\mathbf{n} = -(\mathbf{j}\sin x + \mathbf{k}\cos x)$, $\rho = 2$.

126. $\mathbf{t} = \dfrac{\mathbf{i}\cos x + \mathbf{j}\sin x + \mathbf{k}}{\sqrt{2}}$, $\mathbf{n} = \pm(\mathbf{j}\cos x - \mathbf{i}\sin x)$, $\rho = 2\,|\sec x\,|$.

127. $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$, $\rho = \frac{3}{2}\sqrt{2}$.

128. $\sqrt{2a^2 + 1}\,(e - 1)$.
129. $\frac{1}{2}a \ln 3$.
130. $\dfrac{\pi}{2} - \theta$.
131. $2a\omega\mathbf{u}_\theta - \omega^2 r\mathbf{u}_r$.

132. $\mathbf{v} = \mathbf{u}_r + \mathbf{k}$, $\mathbf{a} = 2\mathbf{u}_\theta$, $\rho = 1$.

133. $\mathbf{v} = -6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$, $\dfrac{d\mathbf{i}}{dt} = 2\mathbf{k} + 4\mathbf{j}$, $\dfrac{d\mathbf{j}}{dt} = \mathbf{k} - 4\mathbf{i}$, $\dfrac{d\mathbf{k}}{dt} = -2\mathbf{i} - \mathbf{j}$.

135. $\dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d^2\mathbf{r}_0}{dt^2} + \mathbf{a} + 2\boldsymbol{\omega} \times \mathbf{v}$.

Chapter XVI. Pages 429–436

1. $2(x - y),\ 2(4y - x)$.
2. $y \cos (xy),\ x \cos (xy)$.
3. $-\dfrac{1}{x}, \dfrac{1}{y}$.
4. $-\dfrac{2y}{(x - y)^2}, \dfrac{2x}{(x - y)^2}$.
5. $2x - 3y + 2z$.
6. $3 \cos (2x + 3y + z)$.

Chapter XVI (Continued)

7. $\dfrac{3}{z}$.

8. $\dfrac{x}{\sqrt{x^2+y^2+z^2}}$.

9. $3(x^2+y^2-2xy-z^2)$.

10. 5.

11. 2.

14. 1.

15. r^2.

16. $\dfrac{b^3}{c^3}$.

17. $\dfrac{c^2\sin^2 B}{2\sin^2(A+B)}$.

18. $\dfrac{bc}{a}\sin A$.

19. $(1, -2, 3)$.

20. $(-1, 3, 10)$.

22. $(2x-y)\,dx+(6y-x)\,dy$.

23. $\dfrac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}}$.

24. $2xy^3z^4\,dx+3x^2y^2z^4\,dy+4x^2y^3z^3\,dz$.

25. $\dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{z}$.

26. $dz=-0.15$, $\Delta z=-0.1495$.

27. $du=0.7$, $\Delta u=0.688$.

28. $dv=54$ in.^{3}, $\Delta v=55.64$ in.^{3}

30. 0.03 ft.

31. 0.006 radian.

32. 0.07.

33. 0.

34. 0.302 ft.

39. $\sin(x+y)\dfrac{\partial z}{\partial x}+z\cos(x+y)$.

40. $2x+z+(x-2z)\dfrac{\partial z}{\partial x}$.

41. $ye^{x-z}\left(1-\dfrac{\partial z}{\partial x}\right)$.

42. $\dfrac{1}{y+z}\dfrac{\partial z}{\partial x}$.

43. $\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}$.

44. $\dfrac{\partial f}{\partial x}+2\dfrac{\partial f}{\partial y}$.

45. $\dfrac{\partial f}{\partial x}+2x\dfrac{\partial f}{\partial z}$.

46. $\dfrac{\partial F}{\partial x}+\dfrac{\partial F}{\partial y}\phi'(x)+\dfrac{\partial F}{\partial z}\left[\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}\phi'(x)\right]$.

47. $\dfrac{\partial F}{\partial x}-\dfrac{\partial F}{\partial v}$.

58. $-\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}$.

59. $-\frac{2}{3}\sqrt{3}$.

60. $\dfrac{\dfrac{\partial f}{\partial y}-\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}+\dfrac{\partial f}{\partial z}}$.

62. $\dfrac{u}{u^2+v^2}$.

63. $\dfrac{u+v}{1+uy}$.

64. $\dfrac{z}{z-x}$.

65. $\dfrac{y-x}{y}$.

68. $(x-1)+2(y-2)+2(z-2)=0$, $x-1=\dfrac{y-2}{2}=\dfrac{z-2}{2}$.

69. $3x+y=8$, normal $x-3y=6$, $z=1$.

70. $3x+4y-5z=0$, $\dfrac{x-3}{3}=\dfrac{y-4}{4}=\dfrac{z-5}{-5}$.

Chapter XVI (Continued)

71. $x + 3y - z = 5,\ x - 1 = \dfrac{y-3}{3} = \dfrac{z-5}{-1}.$

72. $3x - 2y + z = 6,\ \dfrac{x-2}{3} = \dfrac{y+3}{-2} = z + 6.$

73. $2y + 3z - 2x = 1, \dfrac{x-2}{2} = \dfrac{y+2}{-2} = \dfrac{z-3}{-3}.$

74.
$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 - r\sin^2\phi\left(\frac{d\theta}{dt}\right)^2,$$
$$a_\phi = r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} - r\sin\phi\cos\phi\left(\frac{d\theta}{dt}\right)^2,$$
$$a_\theta = r\sin\phi\frac{d^2\theta}{dt^2} + 2r\cos\phi\frac{d\theta}{dt}\frac{d\phi}{dt} + 2\sin\phi\frac{dr}{dt}\frac{d\theta}{dt}.$$

75.
$$a_r = -a\left(\frac{d\phi}{dt}\right)^2 - a\sin^2\phi\left(\frac{d\theta}{dt}\right)^2,\quad a_\phi = a\frac{d^2\phi}{dt^2} - a\sin\phi\cos\phi\left(\frac{d\theta}{dt}\right)^2,$$
$$a_\theta = a\sin\phi\frac{d^2\theta}{dt^2} + 2a\cos\phi\frac{d\theta}{dt}\frac{d\phi}{dt}.$$

78. $\mathbf{u}_\theta\dfrac{d^2\phi}{dt^2} + \mathbf{k}\dfrac{d^2\theta}{dt^2} - (\mathbf{u}_r\sin\phi + \mathbf{u}_\phi\cos\phi)\dfrac{d\theta}{dt}\dfrac{d\phi}{dt}.$

79. $(0, 0, \pm 1)$.

80. The altitude of the cone is twice that of the cylinder.

83. $(3, 3, 9)$.

89.
$$\frac{2\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial x\,\partial y} - \left(\dfrac{\partial f}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial f}{\partial y}\right)^2\dfrac{\partial^2 f}{\partial x^2}}{\left(\dfrac{\partial f}{\partial y}\right)^3}.$$

90. $2\dfrac{\partial f}{\partial y} + \dfrac{\partial^2 f}{\partial x^2} + 4x\dfrac{\partial^2 f}{\partial x\,\partial y} + 4x^2\dfrac{\partial^2 f}{\partial y^2}.$

91. $(x^2 + y^2)\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right).$

92. $2u\left(\dfrac{\partial^2 F}{\partial u^2} + \dfrac{\partial^2 F}{\partial v^2}\right) + 4v\dfrac{\partial^2 F}{\partial u\,\partial v} + 2\dfrac{\partial F}{\partial u}$

93. $\dfrac{\partial^2 F}{\partial u\,\partial v}.$

94. $\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2}.$

95. $A_{mn} = a^{-m}b^{-n}.$

96. $A_{mn} = a^{-m}b^{-n}\dfrac{(m+n)!}{m!n!}.$

97. Not exact.
98. Exact.
99. Not exact.
100. Not exact.
101. Exact.
102. Exact.
103. Exact.
104. 1.
105. -1.

106. $-\dfrac{\partial f}{\partial x}.$

107. $\mathbf{i}e^z + \mathbf{j}e^z + \mathbf{k}(x + y)e^z.$

108. $\sqrt{30}$.

109. $\dfrac{1}{r}\mathbf{u}_\theta.$

110. -1.

111. 224.
112. $\frac{4}{3}$.
113. 31.5.
114. $\frac{2}{3}\pi$.
115. 1.

Chapter XVII. Pages 458–463

1. 2.
2. $\frac{1}{6}\ln 2$.
3. $\frac{1}{4}a$.
4. 4.
5. 6.
6. $\frac{256}{15}$.
11. $\int_0^4\int_0^{\frac{1}{2}y} f(x, y)\,dx\,dy$.
12. $\int_0^{\sqrt{2}}\int_0^{\sqrt{1-\frac{x^2}{2}}} f(x, y)\,dy\,dx$.
13. $\int_2^4\int_{\frac{1}{4}y^2}^4 f(x, y)\,dx\,dy$.
14. $\int_0^4\int_{\frac{1}{2}x}^{x^{\frac{1}{2}}} f(x, y)\,dy\,dx$.
15. $\bar{x} = 1$, $\bar{y} = 0$.
16. $\bar{x} = \frac{6}{5}$, $\bar{y} = -1$.
17. $\frac{344}{105}a^4$.
18. $\frac{1}{6}M(a^2 + b^2)$.
19. $\frac{2}{3}Ma^2$.
20. 6.
21. $\frac{8}{3}$.
22. 8.
23. $\frac{1}{6}$.
24. 4.
25. $\frac{80}{3}$.
26. $\frac{8}{15}a^3$.
27. $\frac{1}{8}\pi a^4$.
28. $\frac{1}{8}\pi a^2$.
29. $\frac{1}{6}\pi a^3$.
30. $\frac{1}{6}a^3[\sqrt{2} + \ln(1 + \sqrt{2})]$.
31. $\frac{1}{4}\pi$.
32. $\sqrt{2} - 1$.
33. $\frac{a^2}{48}(2\pi + 3\sqrt{3})$.
34. $\frac{1}{4}a^2\sqrt{3}$.
35. $\bar{x} = \frac{20}{11}$, $\bar{y} = 0$.
36. $\bar{x} = \dfrac{(3\pi + 8)a}{6\pi + 12}$, $\bar{y} = \dfrac{a}{\pi + 2}$.
37. $\bar{x} = \dfrac{(\pi + 2)a}{8}$, $\bar{y} = \dfrac{(14 - 3\pi)a}{24}$.
38. $\frac{3}{2}Ma^2$.
39. $\frac{1}{8}\pi a^4$.
40. $\frac{1}{8}a^4(2\alpha - \sin 2\alpha)$.
41. $\frac{1}{4}\pi a^4$.
42. $\frac{2}{3}\pi a^3(1 - \cos\alpha)$.
43. $\frac{11}{12}\pi a^3$.
44. $\frac{1}{4}M(4b^2 + 3a^2)$.
45. 27ρ.
46. $\frac{4}{15}$.
47. $\frac{3}{4}\pi a^3$.
48. $\frac{4}{3}$.
49. $(\frac{3}{8}a, \frac{3}{8}b, \frac{3}{8}c)$.
50. $\frac{4}{15}\pi abc(a^2 + b^2)$.
51. $4\pi\sqrt{2}$.
52. $\frac{4}{3}\pi(8 - 3\sqrt{3})a^3$.
53. $\frac{2}{5}Ma^2$.
54. $\frac{3}{20}M(a^2 + 4h^2)$.
55. $\frac{7}{5}Ma^2$.
56. $\dfrac{8\pi\rho a^5}{45}$.
57. $\pi\rho\left[\frac{4}{15}(b^5 - a^5) - \frac{c}{2}(b^4 - a^4) + \frac{c^3}{3}(b^2 - a^2)\right]$.
58. $\frac{2}{3}\pi(5\sqrt{5} - 4)$.
59. $\frac{32}{9}a^3$.
60. $\frac{1}{3}\pi a^3$, πa^3.
61. $\frac{9}{2}\pi a^3$, $\frac{37}{6}\pi a^3$.
62. $\frac{1}{2}\pi a^3$.
63. $\frac{3}{4}\pi a^3$.
64. $\frac{1}{48}a^3(3\pi + 4)$.
65. π.
66. $\dfrac{\pi}{2}$.
67. $\frac{3}{5}Ma^2$.
68. On the axis at distance $\frac{2}{5}h$ from the base.
69. On the axis at distance $\frac{1}{5}h$ from the base.

Chapter XVII (Continued)

70. On the x-axis at distance $\frac{9}{16}a$ from the center of the sphere.
71. On the axis of the cone at distance $\frac{3}{8}a(1 + \cos \alpha)$ from the vertex.

72. $\dfrac{67\pi a^5}{480}$.
73. $\frac{4}{9}a^3$.
74. $\frac{1}{4}\pi^2 a^3$.
75. $\frac{1}{3}\pi[2(b^3 - a^3) - 3c(b^2 - a^2)]$.

76. On the radius perpendicular to the plane face at distance $\frac{3}{8}a$ from the center.
77. On the axis at distance $\frac{4}{5}h$ from the vertex.

78. $8ka^5$.
79. $\bar{x} = \bar{y} = \bar{z} = \frac{7}{12}a$.
80. $\frac{1}{3}Ma^2$.
81. $\frac{1}{3}a^3$.
82. $q^2 + \frac{1}{2}a^2$.
83. $\frac{1}{3}a[\sqrt{2} + \ln(1 + \sqrt{2})]$.
84. $\frac{1}{4}a^2$.
85. $\frac{2}{5}a$.
86. $\dfrac{4a}{\pi}$.
87. $3\sqrt{14}$.
88. $\pi a^2\sqrt{3}$.
89. $\pi a^2\sqrt{2}$.
90. $\frac{2}{3}\pi a^2(3\sqrt{3} - 1)$.
91. 4.
92. $2\pi\sqrt{6}$.
93. $\pi\sqrt{15}$.
94. $\frac{1}{2}\pi a^2\sqrt{2}$.
95. $\frac{4}{3}a$.
96. $\dfrac{1}{a}$.

TABLE OF INTEGRALS

1. $\int u^n\,du = \frac{u^{n+1}}{n+1}$, if n is not -1.

2. $\int \frac{du}{u} = \ln u$.

3. $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$.

4. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\frac{u-a}{u+a}$.

5. $\int e^u\,du = e^u$.

6. $\int a^u\,du = \frac{a^u}{\ln a}$.

Integrals of Trigonometric Functions

7. $\int \sin u\,du = -\cos u$.

8. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u = \frac{1}{2}(u - \sin u \cos u)$.

9. $\int \sin^4 u\,du = \frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u$.

10. $\int \sin^6 u\,du = \frac{5}{16}u - \frac{1}{4}\sin 2u + \frac{1}{48}\sin^3 2u + \frac{3}{64}\sin 4u$.

11. $\int \cos u\,du = \sin u$.

12. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u = \frac{1}{2}(u + \sin u \cos u)$.

13. $\int \cos^4 u\,du = \frac{3}{8}u + \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u$.

14. $\int \cos^6 u\,du = \frac{5}{16}u + \frac{1}{4}\sin 2u - \frac{1}{48}\sin^3 2u + \frac{3}{64}\sin 4u$.

15. $\int \tan u\,du = -\ln\cos u$.

16. $\int \cot u\,du = \ln\sin u$.

17. $\int \sec u\, du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2} + \frac{\pi}{4}\right).$

18. $\int \sec^2 u\, du = \tan u.$

19. $\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln(\sec u + \tan u).$

20. $\int \csc u\, du = \ln(\csc u - \cot u) = \ln \tan \frac{u}{2}.$

21. $\int \csc^2 u\, du = -\cot u.$

22. $\int \csc^3 u\, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln(\csc u - \cot u).$

Integrals containing $\sqrt{a^2 - u^2}$

23. $\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}.$

24. $\int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a}.$

25. $\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} + a\ln\frac{a - \sqrt{a^2 - u^2}}{u}.$

26. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}.$

27. $\int \frac{u^2\, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}.$

28. $\int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a}\ln\frac{a - \sqrt{a^2 - u^2}}{u}.$

29. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}.$

30. $\int (a^2 - u^2)^{\frac{3}{2}}\, du = \frac{u}{8}(5a^2 - 2u^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a}.$

31. $\int \frac{du}{(a^2 - u^2)^{\frac{3}{2}}} = \frac{u}{a^2\sqrt{a^2 - u^2}}.$

Integrals containing $\sqrt{u^2 - a^2}$

32. $\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left(u + \sqrt{u^2 - a^2}\right).$

33. $\int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left(u + \sqrt{u^2 - a^2}\right).$

34. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a \sec^{-1} \frac{u}{a}.$

35. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}}\, du = \ln (u + \sqrt{u^2 - a^2}).$

36. $\displaystyle\int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2} \sqrt{u^2 - a^2} + \frac{a^2}{2} \ln (u + \sqrt{u^2 - a^2}).$

37. $\displaystyle\int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a}.$

38. $\displaystyle\int \frac{du}{u^2 \sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u}.$

39. $\displaystyle\int (u^2 - a^2)^{\frac{3}{2}} du = \frac{u}{8} (2\, u^2 - 5\, a^2) \sqrt{u^2 - a^2} + \frac{3\, a^4}{8} \ln (u + \sqrt{u^2 - a^2}).$

40. $\displaystyle\int \frac{du}{(u^2 - a^2)^{\frac{3}{2}}} = - \frac{u}{\sqrt{u^2 - a^2}}.$

Integrals containing $\sqrt{u^2 + a^2}$

41. $\displaystyle\int \sqrt{u^2 + a^2}\, du = \frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln (u + \sqrt{u^2 + a^2}).$

42. $\displaystyle\int u^2 \sqrt{u^2 + a^2}\, du = \frac{u}{8} (2\, u^2 + a^2) \sqrt{u^2 + a^2} - \frac{a^4}{8} \ln (u + \sqrt{u^2 + a^2}).$

43. $\displaystyle\int \frac{\sqrt{u^2 + a^2}}{u}\, du = \sqrt{u^2 + a^2} + a \ln \frac{\sqrt{u^2 + a^2} - a}{u}.$

44. $\displaystyle\int \frac{du}{\sqrt{u^2 + a^2}} = \ln (u + \sqrt{u^2 + a^2}).$

45. $\displaystyle\int \frac{u^2\, du}{\sqrt{u^2 + a^2}} = \frac{u}{2} \sqrt{u^2 + a^2} - \frac{a^2}{2} \ln (u + \sqrt{u^2 + a^2}).$

46. $\displaystyle\int \frac{du}{u \sqrt{u^2 + a^2}} = \frac{1}{a} \ln \frac{\sqrt{u^2 + a^2} - a}{u}.$

47. $\displaystyle\int \frac{du}{u^2 \sqrt{u^2 + a^2}} = - \frac{\sqrt{u^2 + a^2}}{a^2 u}.$

48. $\displaystyle\int (u^2 + a^2)^{\frac{3}{2}}\, du = \frac{u}{8} (2\, u^2 + 5\, a^2) \sqrt{u^2 + a^2} + \frac{3\, a^4}{8} \ln (u + \sqrt{u^2 + a^2}).$

49. $\displaystyle\int \frac{du}{(u^2 + a^2)^{\frac{3}{2}}} = \frac{u}{a^2 \sqrt{u^2 + a^2}}.$

Other Integrals

50. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$

51. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2}.$

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N	0	1	2	3	4	5	6	7	8	9
0	......	0.0000	0.6931	1.0986	1.3863	1.6094	1.7918	1.9459	2.0794	2.1972
1	2.3026	2.3979	2.4849	2.5649	2.6391	2.7081	2.7726	2.8332	2.8904	2.9444
2	9957	3.0445	3.0910	3.1355	3.1781	3.2189	3.2581	3.2958	3.3322	3.3673
3	3.4012	4340	4657	4965	5264	5553	5835	6109	6376	6636
4	6889	7136	7377	7612	7842	8067	8286	8501	8712	8918
5	9120	9318	9512	9703	9890	4.0073	4.0254	4.0431	4.0604	4.0775
6	4.0943	4.1109	4.1271	4.1431	4.1589	1744	1897	2047	2195	2341
7	2485	2627	2767	2905	3041	3175	3307	3438	3567	3694
8	3820	3944	4067	4188	4308	4427	4543	4659	4773	4886
9	4998	5109	5218	5326	5433	5539	5643	5747	5850	5951
10	6052	6151	6250	6347	6444	6540	6634	6728	6821	6913
11	7005	7095	7185	7274	7362	7449	7536	7622	7707	7791
12	7875	7958	8040	8122	8203	8283	8363	8442	8520	8598
13	8675	8752	8828	8903	8978	9053	9127	9200	9273	9345
14	9416	9488	9558	9628	9698	9767	9836	9904	9972	5.0039
15	5.0106	5.0173	5.0239	5.0304	5.0370	5.0434	5.0499	5.0562	5.0626	0689
16	0752	0814	0876	0938	0999	1059	1120	1180	1240	1299
17	1358	1417	1475	1533	1591	1648	1705	1761	1818	1874
18	1930	1985	2040	2095	2149	2204	2257	2311	2364	2417
19	2470	2523	2575	2627	2679	2730	2781	2832	2883	2933
20	2983	3033	3083	3132	3181	3230	3279	3327	3375	3423
21	3471	3519	3566	3613	3660	3706	3753	3799	3845	3891
22	3936	3982	4027	4072	4116	4161	4205	4250	4293	4337
23	4381	4424	4467	4510	4553	4596	4638	4681	4723	4765
24	4806	4848	4889	4931	4972	5013	5053	5094	5134	5175
25	5215	5255	5294	5334	5373	5413	5452	5491	5530	5568
26	5607	5645	5683	5722	5759	5797	5835	5872	5910	5947
27	5984	6021	6058	6095	6131	6168	6204	6240	6276	6312
28	6348	6384	6419	6454	6490	6525	6560	6595	6630	6664
29	6699	6733	6768	6802	6836	6870	6904	6937	6971	7004
30	7038	7071	7104	7137	7170	7203	7236	7268	7301	7333
31	7366	7398	7430	7462	7494	7526	7557	7589	7621	7652
32	7683	7714	7746	7777	7807	7838	7869	7900	7930	7961
33	7991	8021	8051	8081	8111	8141	8171	8201	8230	8260
34	8289	8319	8348	8377	8406	8435	8464	8493	8522	8551
35	8579	8608	8636	8665	8693	8721	8749	8777	8805	8833
36	8861	8889	8916	8944	8972	8999	9026	9054	9081	9108
37	9135	9162	9189	9216	9243	9269	9296	9322	9349	9375
38	9402	9428	9454	9480	9506	9532	9558	9584	9610	9636
39	9661	9687	9713	9738	9764	9789	9814	9839	9865	9890
40	9915	9940	9965	9989	6.0014	6.0039	6.0064	6.0088	6.0113	6.0137
41	6.0162	6.0186	6.0210	6.0234	0259	0283	0307	0331	0355	0379
42	0403	0426	0450	0474	0497	0521	0544	0568	0591	0615
43	0638	0661	0684	0707	0730	0753	0776	0799	0822	0845
44	0868	0890	0913	0936	0958	0981	1003	1026	1048	1070
45	1092	1115	1137	1159	1181	1203	1225	1247	1269	1291
46	1312	1334	1356	1377	1399	1420	1442	1463	1485	1506
47	1527	1549	1570	1591	1612	1633	1654	1675	1696	1717
48	1738	1759	1779	1800	1821	1841	1862	1883	1903	1924
49	1944	1964	1985	2005	2025	2046	2066	2086	2106	2126
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51	2344	2364	2383	2403	2422	2442	2461	2480	2500	2519
52	2538	2558	2577	2596	2615	2634	2653	2672	2691	2710
53	2729	2748	2766	2785	2804	2823	2841	2860	2879	2897
54	2916	2934	2953	2971	2989	3008	3026	3044	3063	3081
55	3099	3117	3135	3154	3172	3190	3208	3226	3244	3261
56	3279	3297	3315	3333	3351	3368	3386	3404	3421	3439
57	3456	3474	3491	3509	3526	3544	3561	3578	3596	3613
58	3630	3648	3665	3682	3699	3716	3733	3750	3767	3784
59	3801	3818	3835	3852	3869	3886	3902	3919	3936	3953
60	3969	3986	4003	4019	4036	4052	4069	4085	4102	4118
61	4135	4151	4167	4184	4200	4216	4232	4249	4265	4281
62	4297	4313	4329	4345	4362	4378	4394	4409	4425	4441
63	4457	4473	4489	4505	4520	4536	4552	4568	4583	4599
64	4615	4630	4646	4661	4677	4693	4708	4723	4739	4754
65	4770	4785	4800	4816	4831	4846	4862	4877	4892	4907
66	4922	4938	4953	4968	4983	4998	5013	5028	5043	5058
67	5073	5088	5103	5117	5132	5147	5162	5177	5191	5206
68	5221	5236	5250	5265	5280	5294	5309	5323	5338	5352
69	5367	5381	5396	5410	5425	5439	5453	5468	5482	5497
70	5511	5525	5539	5554	5568	5582	5596	5610	5624	5639
71	5653	5667	5681	5695	5709	5723	5737	5751	5765	5779
72	5793	5806	5820	5834	5848	5862	5876	5889	5903	5917
73	5930	5944	5958	5971	5985	5999	6012	6026	6039	6053
74	6067	6080	6093	6107	6120	6134	6147	6161	6174	6187
75	6201	6214	6227	6241	6254	6267	6280	6294	6307	6320
76	6333	6346	6359	6373	6386	6399	6412	6425	6438	6451
77	6464	6477	6490	6503	6516	6529	6542	6554	6567	6580
78	6593	6606	6619	6631	6644	6657	6670	6682	6695	6708
79	6720	6733	6746	6758	6771	6783	6796	6809	6821	6834
80	6846	6859	6871	6884	6896	6908	6921	6933	6946	6958
81	6970	6983	6995	7007	7020	7032	7044	7056	7069	7081
82	7093	7105	7117	7130	7142	7154	7166	7178	7190	7202
83	7214	7226	7238	7250	7262	7274	7286	7298	7310	7322
84	7334	7346	7358	7370	7382	7393	7405	7417	7429	7441
85	7452	7464	7476	7488	7499	7511	7523	7534	7546	7558
86	7569	7581	7593	7604	7616	7627	7639	7650	7662	7673
87	7685	7696	7708	7719	7731	7742	7754	7765	7776	7788
88	7799	7811	7822	7833	7845	7856	7867	7878	7890	7901
89	7912	7923	7935	7946	7957	7968	7979	7991	8002	8013
90	8024	8035	8046	8057	8068	8079	8090	8101	8112	8123
91	8134	8145	8156	8167	8178	8189	8200	8211	8222	8233
92	8244	8255	8265	8276	8287	8298	8309	8320	8330	8341
93	8352	8363	8373	8384	8395	8405	8416	8427	8437	8448
94	8459	8469	8480	8491	8501	8512	8522	8533	8544	8554
95	8565	8575	8586	8596	8607	8617	8628	8638	8648	8659
96	8669	8680	8690	8701	8711	8721	8732	8742	8752	8763
97	8773	8783	8794	8804	8814	8824	8835	8845	8855	8865
98	8876	8886	8896	8906	8916	8926	8937	8947	8957	8967
99	8977	8987	8997	9007	9017	9027	9037	9048	9058	9068
100	9078	9088	9098	9108	9117	9127	9137	9147	9157	9167
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